

Diversity of MIMO Multihop Relay Channels

Sheng Yang and Jean-Claude Belfiore

Abstract

We consider *slow* fading relay channels with a single multi-antenna source-destination terminal pair. The source signal arrives at the destination via N hops through $N - 1$ layers of relays. We analyze the diversity of such channels with *fixed* network size at *high SNR*. In the clustered case where the relays within the same layer can have full cooperation, the cooperative decode-and-forward (DF) scheme is shown to be optimal in terms of the diversity-multiplexing tradeoff (DMT). The upper bound on the DMT, the cut-set bound, is attained. In the non-clustered case, we show that the naive amplify-and-forward (AF) scheme has the maximum multiplexing gain of the channel but is suboptimal in diversity, as compared to the cut-set bound. To improve the diversity, space-time relay processing is introduced through the parallel partition of the multihop channel. The idea is to let the source signal go through K different “AF paths” in the multihop channel. This *parallel AF scheme* creates a parallel channel in the time domain and has the maximum diversity if the partition is properly designed. Since this scheme does not achieve the maximum multiplexing gain in general, we propose a *flip-and-forward* (FF) scheme that is built from the parallel AF scheme. It is shown that the FF scheme achieves both the maximum diversity and multiplexing gains in a distributed multihop channel of arbitrary size. In order to realize the DMT promised by the relaying strategies, approximately universal coding schemes are also proposed.

Index Terms

Relay channel, multiple-input multiple-output (MIMO), multihop, diversity-multiplexing tradeoff (DMT), amplify-and-forward (AF).

Manuscript submitted to the IEEE Transactions on Information Theory. The authors are with the Department of Communications and Electronics, École Nationale Supérieure des Télécommunications, 46, rue Barrault, 75013 Paris, France (e-mail: syang@enst.fr; belfiore@enst.fr).

Diversity of MIMO Multihop Relay Channels

I. INTRODUCTION

Recent years have seen a surge of interest in wireless networks. Unlike the traditional point-to-point communication, elementary modes of cooperation such as relaying are needed to improve both the throughput and reliability in a wireless network. Although capacity of a relay channel [1], [2] is still unknown in general, considerable progress has been made on several aspects, including some achievable capacity results [3], [4] and capacity scaling laws of large networks [5]–[9]. In parallel, research on the cooperative diversity [10], [11], where the relays help the source exploit the spatial diversity of a slow fading channel in a distributed fashion, has attracted significant attention [12]–[18].

In small relay networks where the source signal can reach the destination terminal via a direct link, many results have been known in both the channel capacity [2], [3] and the cooperative diversity. The capacity results are mostly based on the decode-and-forward (DF) and the compress-and-forward (CF) strategies. The amplify-and-forward (AF) scheme, however, is rarely considered in this scenario due to the noise accumulation at the relays. On the other hand, the AF scheme is widely used for cooperative diversity. It has been shown in [13], [15] that the AF scheme is as good as the DF scheme at high SNR as far as the diversity is concerned. Furthermore, it is pointed out in [17] that not needing to decode the source signal makes the relays more capable of protecting the source signal in some cases. The CF scheme, which works with perfect global channel state information (CSI), is usually excluded in the cooperative diversity scenario for practical considerations. In larger relay networks, where direct source-destination links are generally absent, substantial results on the capacity scaling laws have been obtained in the large network size regime [5]–[7], [9]. However, much less is known about the cooperative diversity than in the case of small networks.

This paper analyzes the cooperative diversity in relay networks with a single multi-antenna source-destination terminal pair. The source signal arrives at the destination via a sequence of N hops through $N - 1$ layers of relays. Similar channel setting with a single layer has been studied in [19]–[21] in different contexts. Using large random matrix theory, the ergodic capacity results of some particular relaying schemes have been established for large networks [19]. Recently,

the study has been extended to the case with multiple layers of relays [22] and the case with multiple source-destination pairs [8]. Cooperative diversity in this setting was first studied in [20] for the single-antenna case then in [21] for the multi-antenna case, with distributed space-time coding. All the mentioned works assume linear processing at the relays and the DF scheme is not considered. Actually, one can figure out immediately that the DF scheme is not suitable for the multi-antenna setting due to the suboptimality in terms of degrees of freedom. Requiring the relays to decode the source signal restricts the achievable degrees of freedom. This is one of the fundamental differences between the large networks and small networks : the degrees of freedom of the latter are determined by the source-destination link and not by the relaying strategy.

In this work, we suppose that the network size is arbitrary (but fixed) and the signal-to-noise ratio (SNR) is large. The multihop channel is investigated in terms of the diversity-multiplexing tradeoff (DMT). The DMT was introduced in [23] for the point-to-point multi-antenna (MIMO) channels to capture the fundamental tradeoff between the throughput and reliability in a slow fading channel at high SNR. It was then extensively used in multiuser channels such as the multiple access channels [24] and the relay channels [12], [13], [16]–[18] as performance measure and design criterion of different schemes. Our main contributions are summarized in the following paragraphs.

First, we use the information theoretic cut-set bound [25] to derive an upper bound on the DMT of any relaying strategy. In the clustered case where the relays in the same layer can fully cooperate, this bound is shown to be tight. An optimal scheme is the cooperative DF scheme, where the clustered relays perform joint decoding and joint re-encoding.

While the clustered channel is equivalent to a series-channel and does not feature the distributed nature of wireless networks, the non-clustered case is studied as the main focus of the paper. Since no within-layer cooperation is considered, linear processing at the relays is assumed. We start by the AF strategy, which seems to be the natural first choice as a linear relaying scheme. We show that the AF scheme is, in the DMT sense, equivalent to the Rayleigh product (RP) channel, a point-to-point channel whose channel matrix is defined by a product of N Gaussian matrices. That being said, we examine the RP channel in great detail. It turns out that the DMT of a RP channel has a nice recursive structure and lends some intuitive insights into the typical outage events in such channels. The study of the RP channel leads directly to an exact DMT

characterization for the AF scheme in multihop channels of arbitrary size. The closed-form DMT provides simple guidelines on how to efficiently use the available relays with the AF scheme. One such example is how to reduce the number of relays while keeping the same diversity. While the maximum multiplexing gain is achieved, the achievable diversity gain of the AF scheme can be far from maximum diversity gain suggested by the cut-set bound. Specifically, the DMT of the AF scheme is limited by a virtual “bottleneck” channel.

The following question is then raised : is the DMT cut-set bound tight in the non-clustered case? The question is partially answered in this work : there exists a scheme that achieves both extremes of the cut-set bound, that is, the maximum diversity extreme and maximum multiplexing extreme. In order to achieve the maximum diversity gain, the key is space-time relay processing. Noting that the AF scheme is space-only, we incorporate the temporal processing into the AF scheme. The first scheme that we propose is the *parallel AF* scheme. By partitioning the multihop channel into K “AF paths”, we create a set of K parallel sub-channels in the time domain. A packet that goes through the parallel channel attains an improved diversity if the partition is properly designed. It is shown that there is at least one partition such that the maximum diversity is achieved. However, the parallel AF scheme does not have the maximum multiplexing gain in general, since the achievable degrees of freedom by the scheme are restricted by those of the individual AF paths. In most cases, the AF paths are not as “wide” as the original channel in terms of the degrees of freedom. In order to overcome the loss of degrees of freedom, we linearly transform the set of parallel AF channels into another set in which each sub-channel has the same degrees of freedom as the multihop channel. In the new parallel channel, each relay only need to flip the received signal in a pre-assigned mode, hence the name *flip-and-forward* (FF). It is shown that the FF scheme achieves both the maximum diversity and multiplexing gains. Furthermore, the DMT of the FF scheme is lower-bounded by that of the AF scheme.

Using the results obtained in the non-clustered case, we revisit the clustered case by pointing out that the cooperative DF operation might not be needed in all clusters to get the maximum diversity. We also indicate that cross-antenna linear processing in each cluster helps to improve the DMT only when both transmitter CSI and receiver CSI are known to the relays.

Finally, coding schemes are proposed for all the studied relaying strategies. In the clustered case, a series of Perfect space-time block codes (STBCs) [26], [27] with appropriate rates and dimensions are used at the source and each relay cluster that performs the cooperative DF

operation. In the non-clustered case, construction of Perfect STBCs for general parallel MIMO channels is first provided. The constructed codes can be applied directly to the parallel AF scheme and the FF scheme. All suggested coding schemes achieve the DMT despite of the fading statistics and are thus approximately universal [28].

Regarding the notations, we use boldface lower case letters \mathbf{v} to denote vectors, boldface capital letters \mathbf{M} to denote matrices. $\mathcal{CN}(\mu, \sigma^2)$ represents a complex Gaussian random variable with mean μ and variance σ^2 . $\mathbb{E}[\cdot]$ stands for the expectation operator. $[\cdot]^\top, [\cdot]^\dagger$ respectively denote the matrix transposition and conjugated transposition operations. $\|\cdot\|$ is the vector norm. $\|\cdot\|_F$ is the Frobenius matrix norm. We define $\prod_{i=1}^N \mathbf{M}_i \triangleq \mathbf{M}_N \cdots \mathbf{M}_1$ for any matrices \mathbf{M}_i 's. The square root $\sqrt{\mathbf{P}}$ of a positive semi-definite matrix \mathbf{P} is defined as a positive semi-definite matrix such that $\mathbf{P} = \sqrt{\mathbf{P}}(\sqrt{\mathbf{P}})^\dagger$. $\lambda_{\max}(\mathbf{P})$ and $\lambda_{\min}(\mathbf{P})$ denote respectively the maximum and minimum eigenvalues of a semi-definite matrix \mathbf{P} . $(x)^+$ means $\max(0, x)$. $\lceil a \rceil$ (respectively, $\lfloor a \rfloor$) is the closest integer that is not smaller (respectively, not larger) than a . $(a)_b$ means $a \bmod b$. $\log(\cdot)$ stands for the base-2 logarithm. For any quantity q ,

$$q \doteq \text{SNR}^a \quad \text{means} \quad \lim_{\text{SNR} \rightarrow \infty} \frac{\log q}{\log \text{SNR}} = a$$

and similarly for \leq and \geq . The tilde notation $\tilde{\mathbf{n}}$ is used to denote the (increasingly) ordered version of \mathbf{n} . Let \mathbf{m} and \mathbf{n} be two vectors of respective length $L_{\mathbf{m}}$ and $L_{\mathbf{n}}$, then $\mathbf{m} \preceq \mathbf{n}$ means $\tilde{m}_i \leq \tilde{n}_i, \forall i = 1, \dots, \min\{L_{\mathbf{m}}, L_{\mathbf{n}}\} - 1$. $\mathbf{m} \subseteq \mathbf{n}$ means that \mathbf{m} is a sub-vector of some permuted version of \mathbf{n} .

The rest of this paper is organized as follows. Section II describes the system model and some basic assumptions in our work. The DMT cut-set bound and the clustered case with the DF scheme are presented. In Section III, we study the non-clustered case with the AF scheme. The parallel AF and the FF schemes are proposed in Section IV. In section V, the clustered case is revisited. The approximately universal coding schemes are proposed in Section VI. Section VII provides some selected numerical examples. Finally, a brief conclusion is drawn in Section VIII. Most detailed proofs are deferred to the appendices.

II. SYSTEM MODEL AND BASIC ASSUMPTIONS

A. Channel Model

The considered N -hop relay channel model is composed of one source (layer 0), one destination (layer N), and $N - 1$ layers of relays (layer 1 to layer $N - 1$). Each terminal is equipped with multiple antennas. The total number of antennas in layer i is denoted by n_i . For convenience, we define $n_t \triangleq n_0$, $n_r \triangleq n_N$, and $n_{\min} = \min_{i=0,\dots,N} n_i$. We assume that the source signal arrives at the destination via a sequence of N hops through the $N - 1$ layers and that terminals in layer i can only receive the signal from layer $i - 1$. The fading sub-channel between layer $i - 1$ and layer i is denoted by the matrix \mathbf{H}_i . Sub-channels are assumed to be mutually independent, flat Rayleigh-fading and quasi-static. That is, the channel coefficients are independent and identically distributed (i.i.d.) complex circular symmetric Gaussian with unit variance. And they remain constant during a coherence interval of length L and change independently from one coherence interval to another. Furthermore, the transmission is supposed to be perfectly synchronized. Under these assumptions, the signal model within a coherence interval can be written as

$$\mathbf{y}_i[l] = \mathbf{H}_i \mathbf{x}_{i-1}[l] + \mathbf{z}_i[l], \quad l = 1, \dots, L,$$

where $\mathbf{x}_i[l], \mathbf{y}_i[l] \in \mathbb{C}^{n_i \times 1}$ denote the transmitted and received signal at layer i ; $\mathbf{z}_i[l] \in \mathbb{C}^{n_i \times 1}$ is the additive white Gaussian noise (AWGN) at layer i with i.i.d. $\mathcal{CN}(0, 1)$ entries. Since we consider the non-ergodic case where the coherence time interval L is large enough, we drop the time index l hereafter. It is assumed that all relays work in full-duplex¹ mode and the transmission is subject to the short-term power constraint

$$\mathbb{E} \{ \|\mathbf{x}_i\|_F^2 \} \leq \text{SNR}, \quad \forall i \tag{1}$$

with SNR being the average transmitted SNR per layer. All terminals are supposed to have perfect channel state information at the receiver² and no CSI at the transmitter. From now on, we denote the channel as a (n_0, n_1, \dots, n_N) multihop channel.

¹This assumption is merely for simplicity of notation. Since we assume that no cross-talk exists between different channels, the half-duplex constraint is directly translated to a reduction of degrees of freedom by a factor of two and does not impact the relaying strategy. This is achieved by letting all even-numbered (respectively, odd-numbered) nodes transmit (respective, receive) in even-numbered time slot and received (respective, transmit) in odd-numbered time slots.

²As we will see, assuming no CSI at all at the relays will not change the results of our work.

B. Diversity-Multiplexing Tradeoff

Slow fading channels are outage-limited, i.e., there is an *outage probability* $P_{\text{out}}(\text{SNR}, R)$ that the channel cannot support a target data rate of R bits per channel use at signal-to-noise ratio SNR. In the high SNR regime, this fundamental interplay between throughput and reliability is characterized by the diversity-multiplexing tradeoff [23].

Definition 2.1: The *multiplexing gain* r and *diversity gain* d of a fading channel are defined by

$$r \triangleq \lim_{\text{SNR} \rightarrow \infty} \frac{R(\text{SNR})}{\log \text{SNR}} \quad \text{and} \quad d \triangleq - \lim_{\text{SNR} \rightarrow \infty} \frac{\log P_{\text{out}}(\text{SNR}, R)}{\log \text{SNR}}.$$

A more compact form is

$$P_{\text{out}}(\text{SNR}, r \log \text{SNR}) \doteq \text{SNR}^{-d(r)}. \quad (2)$$

Note that in the definition we use the outage probability instead of the error probability, since it is shown in [23] that the error probability of any particular coding scheme with maximum likelihood (ML) decoding is dominated by the outage probability at high SNR and that the thus defined DMT is the best that one can achieve with any coding scheme. In the Rayleigh MIMO channel, the DMT has the following closed form.

Lemma 2.1 ([23]): The DMT of a $n_t \times n_r$ Rayleigh channel is a piecewise-linear function connecting the points $(k, d(k))$, $k = 0, 1, \dots, \min(n_t, n_r)$, where

$$d(k) = (n_t - k)(n_r - k). \quad (3)$$

In the following, we will use the DMT as our performance measure. For convenience of presentation, we provide the following definition.

Definition 2.2: Two channels are said to be *DMT-equivalent* or *equivalent* if they have the same DMT.

C. Upper Bound on the DMT

Before studying any specific relaying strategy, we establish an upper bound on the DMT of the multihop system as a benchmark.

Proposition 2.1 (Cut-set bound): For any relaying strategy \mathcal{T} , we have

$$d^{\mathcal{T}}(r) \leq \bar{d}(r)$$

with

$$\bar{d}(r) \triangleq \min_{i=1,\dots,N} d_i(r), \quad (4)$$

where $d_i(r)$ is the DMT of the point-to-point channel between layer $i - 1$ and layer i . In particular, by defining the maximum diversity gain and multiplexing gain as $d_{\max} \triangleq \bar{d}(0)$ and $r_{\max} \triangleq \sup\{\bar{d}(r) > 0\}$, respectively, we have

$$d_{\max} = \min_{i=1,\dots,N} n_{i-1}n_i, \text{ and} \quad (5)$$

$$r_{\max} = \min_{i=0,\dots,N} n_i. \quad (6)$$

Proof: From the information theoretic cut-set bound [25], the mutual information between the source and the destination satisfies

$$I_{\mathcal{T}}(\mathbf{x}_0; \mathbf{y}_N | \mathbf{H}_1, \dots, \mathbf{H}_N) \leq I(\mathbf{x}_{i-1}; \mathbf{y}_i | \mathbf{H}_i), \quad \forall i,$$

for any relaying strategy \mathcal{T} . Thus, the outage probability using a relaying scheme \mathcal{T} is

$$\begin{aligned} P_{\text{out}}^{\mathcal{T}}(R) &\triangleq \mathbf{P}_{\{\mathbf{H}_i\}_i} \{I_{\mathcal{T}}(\mathbf{x}_0; \mathbf{y}_N | \mathbf{H}_1, \dots, \mathbf{H}_N) < R\} \\ &\geq \max_i \mathbf{P}_{\mathbf{H}_i} \{I(\mathbf{x}_{i-1}; \mathbf{y}_i | \mathbf{H}_i) < R\} \\ &= \max_i P_{\text{out},i}(R), \end{aligned} \quad (7)$$

where $P_{\text{out},i}(R)$ is the outage probability of the i th sub-channel. From (2) and (7), we prove (4). Finally, (5) and (6) are from the direct application of Lemma 2.1. \square

D. The Clustered Case and Decode-and-Forward

If we assume that the relays within the same layer are clustered, i.e., they can perform joint decoding and joint re-coding operations, then each layer can act as a virtual multi-antenna terminal. This could happen either when the relays are controlled by a central unit via wired links or when they are close enough to each other to exchange information perfectly. In this case, the relay channel model is equivalent to a serial concatenation of N independent MIMO channels. Let us consider the following cooperative decode-and-forward scheme. Each layer tries to cooperatively decode the received signal. When a successful decoding is assumed, the embedded message is re-encoded and then forwarded to the next layer. We can show that this simple scheme is DMT optimal.

Proposition 2.2: When the relays are clustered, the cooperative DF scheme achieves the DMT cut-set bound $\bar{d}(r)$ defined in (4).

Proof: To show the achievability, note that the cooperative DF scheme being in outage implies the outage of at least one of the sub-channels. By the union bound,

$$P_{\text{out}}^{\text{DF}}(R) \leq \sum_{i=1}^N P_{\text{out},i}(R).$$

At high SNR, the probability is dominated by the largest term in the sum of the right-hand side (RHS). From (2), we get

$$d^{\text{DF}}(r) \geq \min_{i=1,\dots,N} d_i(r) = \bar{d}(r).$$

□

In the high SNR regime, the union bound defined by the sum operation coincides in the SNR exponent with the cut-set bound defined by the minimum operation. Hence, the DMT cut-set bound is tight in the clustered case. However, relays in wireless networks are not clustered in general. In fact, one of the important and interesting attributes of wireless networks is the distributed nature. In the following two sections, we will concentrate on the non-clustered case and analyze the achievable DMT.

III. AMPLIFY-AND-FORWARD

In this section, we consider the non-clustered case, where the relays work in a distributed manner and no within-layer communication is allowed. In this case, applying the DF scheme at each individual relay might incur loss of degrees of freedom. To see this, take the single-layer channel as an example. In the best case where all the relays succeed in decoding, they transmit the message using a pre-assigned codebook. This scheme transforms the relays-destination channel into a $n_1 \times n_2$ virtual MIMO channel. Before this could possibly happen, however, the success decoding at the relays must be guaranteed with high probability. This constraint imposes that the degrees of freedom in this scheme must not be larger than $\min_k \{n_{1,k}\}$ with $n_{1,k}$ being the number of antennas at the k th relay. While this scheme achieves the maximum multiplexing gain in the single-antenna case, it could fail in the multi-antenna case.

Since we do not know how to cooperate efficiently in this case, we start by the most obvious and naivest relaying scheme : the amplify-and-forward scheme. This scheme in the considered

setting has been studied in [19], [22] for the capacity scaling laws, and in [29] for the DMT. It is worth noting³ that, in [29], a lower bound on the DMT of the AF scheme in a symmetric network ($n_i = n, \forall i$) was obtained, while our work derives the exact DMT for a network of arbitrary dimension with a different approach.

A. Signal Model

In the considered AF scheme, each antenna node normalizes the received signal to the same power level and then retransmits it. This linear operation can be expressed as

$$\mathbf{x}_i = \mathbf{D}_i \mathbf{y}_i, \quad i = 1, \dots, N-1,$$

where, by the power constraint (1),

$$\mathbb{E}(|\mathbf{x}_i(j)|^2) \leq \frac{\text{SNR}}{n_i}, \quad j = 1, \dots, n_i;$$

the scaling matrix $\mathbf{D}_i \in \mathbb{C}^{n_i \times n_i}$ is diagonal due to the antenna-wise nature of the relaying scheme, with the normalization factors⁴

$$\mathbf{D}_i(j, j) = \sqrt{\frac{1}{\frac{\text{SNR}}{n_{i-1}} (\sum_{k=1}^{n_{i-1}} |\mathbf{H}_i(j, k)|^2) + 1}} \cdot \sqrt{\frac{\text{SNR}}{n_i}}. \quad (8)$$

Thus, the signal model of the end-to-end channel is

$$\mathbf{y}_N = \left(\prod_{i=1}^N \mathbf{D}_i \mathbf{H}_i \right) \mathbf{x}_0 + \sum_{j=1}^N \left(\prod_{i=j}^N \mathbf{H}_{i+1} \mathbf{D}_i \right) \mathbf{z}_j, \quad (9)$$

where, for the sake of simplicity, we defined $\mathbf{H}_{N+1} \triangleq \mathbf{I}$ and $\mathbf{D}_N \triangleq \mathbf{I}$. The whitened form of this channel is

$$\mathbf{y} = \sqrt{\mathbf{R}} \left(\prod_{i=1}^N \mathbf{D}_i \mathbf{H}_i \right) \mathbf{x}_0 + \mathbf{z},$$

where \mathbf{z} is the whitened noise and $\sqrt{\mathbf{R}}$ is the whitening matrix with \mathbf{R}^{-1} being the covariance matrix of the noise in (9). Since it can be shown that $\lambda_{\max}(\mathbf{R}) \doteq \lambda_{\min}(\mathbf{R}) \doteq \text{SNR}^0$, \mathbf{R} can be neglected in the DMT analysis and the AF channel⁵ is equivalent to the MIMO channel defined by the following matrix

$$\mathbf{H}_N \mathbf{D}_{N-1} \cdots \mathbf{H}_2 \mathbf{D}_1 \mathbf{H}_1. \quad (10)$$

The rest of the section is devoted to the DMT analysis of this channel.

³The authors found [29] at the very end of the preparation for this manuscript.

⁴In the case where long-term power constraint is imposed, we simply replace the channel coefficients $|\mathbf{H}_i(j, k)|$ in (8) by 1's.

⁵Here, with a slight abuse of terminology, we call the multihop channel with AF scheme an *AF channel*.

B. The Rayleigh Product Channel

Definition 3.1: Let $\mathbf{H}_i \in \mathbb{C}^{n_{i-1} \times n_i}$, $i = 1, 2, \dots, N$, be N independent complex Gaussian matrices with i.i.d. $\mathcal{CN}(0, 1)$ entries. A (n_0, n_1, \dots, n_N) *Rayleigh product* (RP) channel is a $n_N \times n_0$ MIMO channel defined by

$$\mathbf{y} = \sqrt{\frac{\text{SNR}}{n_1 \cdots n_N}} \mathbf{\Pi} \mathbf{x} + \mathbf{z}, \quad (11)$$

where $\mathbf{\Pi} \triangleq \mathbf{H}_1 \mathbf{H}_2 \cdots \mathbf{H}_N$ is the channel matrix; $\mathbf{x} \in \mathbb{C}^{n_N \times 1}$ is the transmitted signal with normalized power, i.e., $\mathbb{E}\{\|\mathbf{x}\|^2\} = n_N$; and $\mathbf{y} \in \mathbb{C}^{n_0 \times 1}$ is the received signal; $\mathbf{z} \in \mathbb{C}^{n_0 \times 1} \sim \mathcal{CN}(0, \mathbf{I})$ is the AWGN; SNR is the SNR per receive antenna. (n_0, n_1, \dots, n_N) is called the *dimension* of the channel and N is called the *length* of the channel.

While this channel model has been studied in terms of the asymptotic eigenvalue distribution in the large dimension regime [30], we are particularly interested in the fixed dimension case in the high SNR regime. In this regime, we can define a more general RP channel as

$$\mathbf{\Pi}_g \triangleq \mathbf{H}_1 \mathbf{T}_{1,2} \mathbf{H}_2 \cdots \mathbf{H}_{N-1} \mathbf{T}_{N-1,N} \mathbf{H}_N. \quad (12)$$

Proposition 3.1: The general RP channel is equivalent to

- a (n_0, n_1, \dots, n_N) RP channel, if all the matrices $\mathbf{T}_{i,i+1}$'s are square and their singular values satisfy $\sigma_j(\mathbf{T}_{i,i+1}) \doteq \text{SNR}^0$, $\forall i, j$;
- a $(n_0, n'_1, \dots, n'_{N-1}, n_N)$ RP channel, with n'_i being the rank of the matrix $\mathbf{T}_{i,i+1}$, if the matrices $\mathbf{T}_{i,i+1}$'s are constant.

Proof: See Appendix II-C. □

Hence, we can consider the RP channel from Definition 3.1 without loss of generality.

1) *Direct Characterization:* Recall that $\tilde{\mathbf{n}}$ is the ordered version of \mathbf{n} with $\tilde{n}_N \geq \tilde{n}_{N-1} \geq \cdots \geq \tilde{n}_0$ and $n_{\min} \triangleq \tilde{n}_0$.

Theorem 3.1: The DMT of a RP channel (n_0, n_1, \dots, n_N) is a piecewise-linear function connecting the points $(k, d^{\text{RP}}(k))$, $k = 0, 1, \dots, n_{\min}$, where

$$d^{\text{RP}}(k) = \sum_{i=k+1}^{n_{\min}} c_i \quad (13)$$

with

$$c_i \triangleq 1 - i + \min_{k=1, \dots, N} \left\lfloor \frac{\sum_{l=0}^k \tilde{n}_l - i}{k} \right\rfloor, \quad i = 1, \dots, n_{\min}. \quad (14)$$

Proof: The DMT depends on the “near zero” probability of the singular values of channel matrix. While this probability for the given product matrix is intractable, we can characterize it by induction on the length N . The main idea is that, conditioned on a given product matrix $\mathbf{H}_1\mathbf{H}_2\cdots\mathbf{H}_{N-1}$, $\mathbf{H}_1\mathbf{H}_2\cdots\mathbf{H}_N$ is Gaussian whose singular distribution is tractable. See Appendix II for details. \square

The following corollaries are given without proofs.

Corollary 3.1 (Permutation invariance): The DMT of a RP channel depends only on the ordered dimension $\tilde{\mathbf{n}}$.

Corollary 3.2 (Monotonicity): The DMT is monotonic in the following senses :

- if $\mathbf{n}_1 \succeq \mathbf{n}_2$, then

$$d_{\mathbf{n}_1}^{\text{RP}}(r) \geq d_{\mathbf{n}_2}^{\text{RP}}(r), \quad \forall r;$$

- if $\mathbf{n}_1 \supseteq \mathbf{n}_2$, then

$$d_{\mathbf{n}_1}^{\text{RP}}(r) \leq d_{\mathbf{n}_2}^{\text{RP}}(r), \quad \forall r.$$

Corollary 3.3 (Symmetric Rayleigh product channels): When $n_0 = \dots = n_N = n$, we have

$$d_n^{\text{RP}}(k) = \frac{(n-k)(n+1-k)}{2} + \frac{a(k)}{2}((a(k)-1)N + 2b(k)), \quad (15)$$

where $a(k) \triangleq \lfloor \frac{n-k}{N} \rfloor$ and $b(k) \triangleq (n-k)_N$.

2) *DMT Equivalent Classes:* Corollary 3.1 implies that RP channels with the same ordered dimension belong to the same DMT equivalent class. In the following, a precise characterization of the DMT class is obtained. Before that, we need the following definitions.

Definition 3.2: A (m_0, m_1, \dots, m_k) RP channel is said to be a *reduction* of a (n_0, n_1, \dots, n_N) RP channel if 1) they are equivalent, 2) $k \leq N$, and 3) $\mathbf{m} \preceq \mathbf{n}$. In particular, if $k = N$, then it is called a *vertical reduction*. Similarly, if $\tilde{m}_i = \tilde{n}_i, \forall i \in [0, k]$, it is a *horizontal reduction*.

Definition 3.3: $(\tilde{n}_0, \tilde{n}_1, \dots, \tilde{n}_{N^*})$ is said to be a *minimal form* if no reduction other than itself exists. Similarly, it is called a *minimal vertical form* (respectively, *minimal horizontal form*) if no vertical (respectively, horizontal) reduction other than itself exists. A RP channel is said to have *order* N^* if its minimal form is of length $N^* + 1$.

Theorem 3.2: A (n_0, n_1, \dots, n_N) RP channel can be reduced to a $(\tilde{n}_0, \tilde{n}_1, \dots, \tilde{n}_k)$ channel if and only if

$$k(\tilde{n}_{k+1} + 1) \geq \sum_{l=0}^k \tilde{n}_l. \quad (16)$$

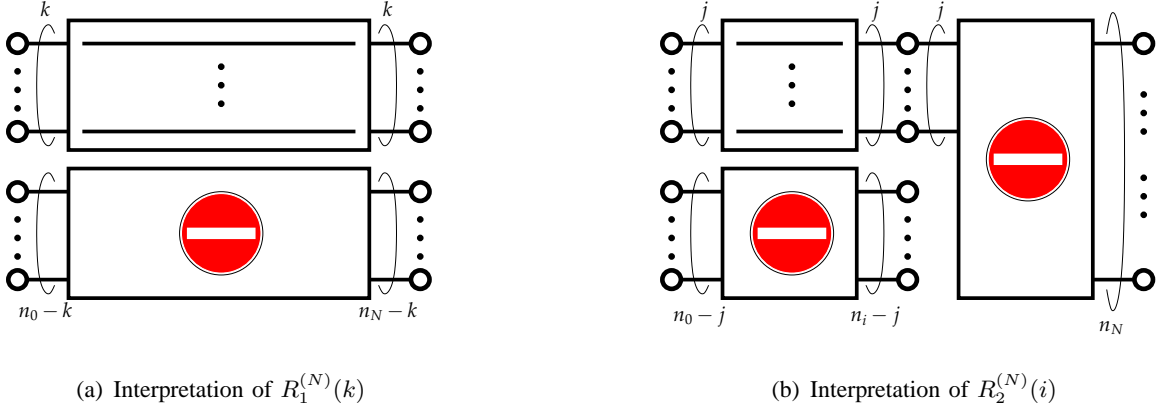


Fig. 1. Interpretations of the DMT of the RP channel.

In particular, it can be reduced to a Rayleigh channel if and only if

$$\tilde{n}_2 + 1 \geq \tilde{n}_0 + \tilde{n}_1. \quad (17)$$

Proof: See Appendix III-A. □

Corollary 3.4: The channel order N^* is the minimum integer such that (16) is satisfied. The minimal horizontal form is the minimal form $(\tilde{n}_0, \tilde{n}_1, \dots, \tilde{n}_{N^*})$ and the minimal vertical form is $(\tilde{n}_0, \tilde{n}_1, \dots, \tilde{n}_{N^*}, \bar{n}, \dots, \bar{n})$ with

$$\bar{n} \triangleq \left\lceil \frac{\sum_{l=0}^{N^*} \tilde{n}_l}{N^*} \right\rceil - 1. \quad (18)$$

For instance, the minimal form of a $(1, n_1, \dots, n_N)$ RP channel is $(1, \tilde{n}_1)$, i.e., a $1 \times \tilde{n}_1$ or $\tilde{n}_1 \times 1$ Rayleigh channel.

Theorem 3.3: The DMT equivalent class is *uniquely* identified by the minimal form, i.e., two RP channels are equivalent *if and only if* they have the same minimal form.

Proof: See Appendix III-B. □

3) *Recursive Characterization:* In order to interpret the closed-form DMT of Theorem 3.1, we derive an equivalent recursive form as shown in the following theorem.

Theorem 3.4: The DMT $d^{\text{RP}}(k)$ defined in (13) can be alternatively characterized by

$$R_1^{(N)}(k) : d_{(n_0, \dots, n_N)}^{\text{RP}}(k) = d_{(n_0-k, \dots, n_N-k)}^{\text{RP}}(0), \quad \forall k; \quad (19)$$

$$R_2^{(N)}(i) : d_{(n_0, \dots, n_N)}^{\text{RP}}(0) = \min_{j \geq 0} \left\{ d_{(n_0, \dots, n_i)}^{\text{RP}}(j) + d_{(j, n_{i+1}, \dots, n_N)}^{\text{RP}}(0) \right\}, \quad \forall i; \quad (20)$$

$$R_3^{(N)}(i, k) : d_{(n_0, \dots, n_N)}^{\text{RP}}(k) = \min_{j \geq k} \left\{ d_{(n_0, \dots, n_i)}^{\text{RP}}(j) + d_{(j, n_{i+1}, \dots, n_N)}^{\text{RP}}(k) \right\}, \quad \forall i, k. \quad (21)$$

Proof: See Appendix IV. □

A new interpretation of the DMT is as follows. *Let us consider a multi-layer network of dimension (n_0, n_1, \dots, n_N) . Then, $d^{\text{RP}}(k)$ is the minimum “cost” to limit the “network flow” between the source and the destination to k (the flow- k event). In particular, the maximum diversity $d^{\text{RP}}(0)$ is the “disconnection cost”.* Now, we can apply the new interpretation to the results of Theorem 3.4. First, $R_1(k)$ says that the most efficient way to limit the flow to k is to keep a (k, k, \dots, k) channel fully connected and to disconnect the $(n_0 - k, n_1 - k, \dots, n_N - k)$ residual channel, as shown in Fig. 1(a). Then, $R_2(i)$ suggests that in order to disconnect a (n_0, n_1, \dots, n_N) channel, if we allow for j flows from the source to some layer i , then the (j, n_{i+1}, \dots, n_N) channel from the j “ends” of the flows at layer i to the destination must be disconnected (Fig. 1(b)). Obviously, the most efficient way is such that the total cost is minimized with respect to j . Finally, the flow- k event takes place when both the flow- j ($j \geq k$) event in the (n_0, \dots, n_i) channel and the flow- k event in the (j, n_{i+1}, \dots, n_N) channel happen at the same time. We can easily verify that $(R_1(k), R_3(i, k))$ is equivalent to $(R_1(k), R_2(i))$. Also note that $R_2(i)$ and $R_3(i, k)$ hold for any layer i , which guarantees the coherence of the interpretation.

The recursive characterization sheds lights on the typical outage event of the RP channel. In the trivial case of $N = 1$ (the Rayleigh channel), the typical and only way for the channel to be in outage at multiplexing gain r approaching to zero is that all the $\tilde{n}_0 \times \tilde{n}_1$ paths are bad, i.e., all channel gains are close to zero. And the disconnection cost is $\tilde{n}_0 \times \tilde{n}_1$. In the non-trivial cases ($N > 1$) where channels are concatenated, there are several types of outage event. Each type is numbered by the index j in (20) and (21). The cost of the type- j event is given by $d_{(n_0, \dots, n_i)}^{\text{RP}}(j) + d_{(j, n_{i+1}, \dots, n_N)}^{\text{RP}}(0)$ for a certain j . Hence, the typical outage event is the one with the minimum cost and it does not necessarily happen when one of the sub-channels being totally bad ($j = 0$ or $j = n_i$). The *mismatch* of two partially bad sub-channels can also cause outage. This phenomenon will be detailed later on.

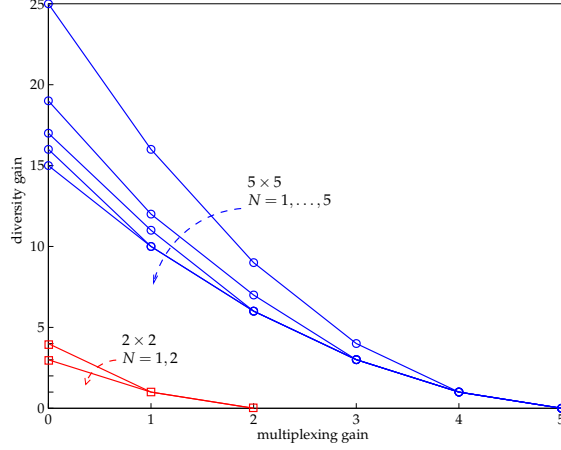


Fig. 2. Diversity-multiplexing tradeoff of 2×2 and 5×5 symmetric RP channels.

C. DMT of the AF Scheme

From the equivalent channel matrix (10) and Proposition 3.1, the AF channel is equivalent to a $(n_N, n_{N-1}, \dots, n_0)$ RP channel.⁶ Therefore, the DMT of the AF channel is

$$d^{\text{AF}}(r) = d^{\text{RP}}(r), \quad \forall r.$$

1) *Implications:* From the results of Section III-B, several interesting implications on the AF scheme with respect to the DMT are summarized below.

- Interchanging layers does not influence the DMT.
- The maximum diversity of the AF scheme is lower- and upper-bounded as

$$\frac{\tilde{n}_0(\tilde{n}_1 + 1)}{2} \leq d_{\max}^{\text{AF}} \leq \tilde{n}_0 \tilde{n}_1 \quad (22)$$

which is obtained via the monotonicity from Corollary 3.2. We have set $\tilde{n}_2 \geq \tilde{n}_0 + \tilde{n}_1 - 1$ for the upper bound and $\tilde{n}_N = \tilde{n}_{N-1} = \dots = \tilde{n}_1$ for the lower bound. The upper bound shows that there exists a virtual $\tilde{n}_0 \times \tilde{n}_1$ “bottleneck” channel that limits the AF scheme and that it is not necessarily one of the sub-channels. On the other hand, the lower bound is always strictly larger than half the upper bound and is independent of the number of hops

⁶Theoretically, this is true only when the singular values $\sigma_j(\mathbf{D}_i) \doteq \text{SNR}^0, \forall i, j$. To this end, it is enough to modify the matrices as $\mathbf{D}_i(j, j) = \min\{\mathbf{D}_i(j, j), \kappa\}$ where $0 < \kappa < \infty$ is a constant independent of SNR. Note that the κ is only for theoretical proof and is not used in practice, since we can always set κ a very large constant but independent of SNR.

N . In the symmetric case (Corollary 3.3), we observe that the DMT degrades with N only when $N \leq n$ and that we have

$$d_{(n,\dots,n)}^{\text{AF}}(k) = \frac{(n-k)(n+1-k)}{2}$$

for $N \geq n$. The observation can also be deduced from theorem 3.2 applying which we infer that the order of any symmetric RP channel with $N > n$ is $N^* = n$. This non-trivial lower bound is somewhat anti-intuition, since it means that at this point introducing extra fading hops does not degrade the diversity any more. An example illustrating the DMT of the 2×2 and 5×5 RP channels of different lengths is in Fig. 2.

- If one could increase the number of antennas at each relay layer without any constraint, then intuition tells us that the AF channel could be reduced to a $n_t \times n_r$ point-to-point Rayleigh MIMO channel and the diversity order is $n_t n_r$. The relay layers “disappear”. The intuition has been confirmed in [19] in the single-layer case with the capacity results. Here, the result in Theorem 3.2 indicates that this happens when there are exactly $n_t + n_r - 1$ antennas at each relay layers from the diversity point of view. Further increasing the number of antennas is not necessary in the DMT sense. On the other hand, if the number of available antennas is fixed, then Corollary 3.4 provides, through the minimal vertical form, the minimum numbers of antennas at each layer to achieve the diversity that could be achieved when all antennas were used. In both cases, our results yield simple guidelines to minimize the number of relay antennas (also the number of relays in general) without loss of optimality of the DMT. In the same way, the number of transmit antennas at the source terminal can also be reduced to lower the coding complexity. A numerical example is given in Section VII.

2) *Comparison to the Cut-Set Bound:* A simple comparison between the DMT of the AF scheme and the cut-set bound (4) is carried out as follows. First, the AF scheme is multiplexing optimal and achieves the maximum multiplexing gain \tilde{n}_0 of the channel. Then, since

$$(\tilde{n}_0 - k)(\tilde{n}_1 - k) \leq \min_{i=1,\dots,N} \{(n_{i-1} - k)(n_i - k)\}, \quad \forall k,$$

the diversity upper bound is generally not achievable by the AF scheme for integer multiplexing gain k . In particular, the best diversity gain of the AF scheme is $\tilde{n}_0 \tilde{n}_1$, while the upper bound is $\min_i \{n_i n_{i+1}\}$. Finally, for any non-integer multiplexing gain, say $r \in (k, k+1)$, $\bar{d}(r)$ is minimum of linear functions and thus concave, while $d^{\text{AF}}(r)$ is linear. The comparison shows

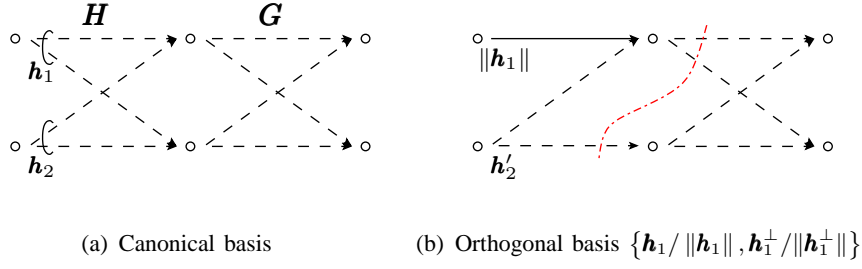


Fig. 3. The (2, 2, 2) multihop channel in two different basis.

that the *bottleneck* of the channel is always one of the hops (inter-layer sub-channels), while the bottleneck of the AF scheme is the virtual $\tilde{n}_0 \times \tilde{n}_1$ channel that does not correspond to any physical sub-channel in most cases. The following remark states the necessary and sufficient condition for the AF scheme to achieve the maximum diversity.

Remark 3.1: The AF scheme achieves the diversity upper bound d_{\max} if and only if it can be reduced to the bottleneck of the channel, i.e.,

$$\min\{n_{i^*}, n_{i^*+1}\} = \tilde{n}_0, \quad \max\{n_{i^*}, n_{i^*+1}\} = \tilde{n}_1, \quad \text{and } \tilde{n}_2 + 1 \geq \tilde{n}_0 + \tilde{n}_1, \quad (23)$$

where i^* is such that $n_i n_{i+1}$ is minimized.

This condition is very stringent. It means that the two layers with minimum numbers of antennas must stand one next to the other and that the other layers must have a large number of antennas. Moreover, note that the AF scheme achieving the maximum diversity does not necessarily mean that it achieves $\bar{d}(r)$ for all r .

3) Mismatch of Adjacent Sub-Channels: In order to achieve the diversity upper bound, intuitively, one should assure that the end-to-end channel is good if each sub-channel is good. However, this property does not hold for the AF scheme that suffers from the *mismatch* of adjacent sub-channels. A concrete example is as follows.

Example 3.1: In the symmetric two-hop channel with $n = 2$ (Fig. 3), the diversity order of the AF scheme is 3 while the upper bound is 4.

Note that the AF channel is in outage if the product channel \mathbf{GH} is bad, i.e., all the singular values of \mathbf{GH} are close to zero. This probability can be decomposed as

$$\mathbf{P}\{\mathbf{GH} \text{ is bad}\} = \mathbf{P}\{\text{both } \mathbf{GH} \text{ and } \mathbf{H} \text{ are bad}\} + \mathbf{P}\{\mathbf{GH} \text{ is bad, while } \mathbf{H} \text{ is not bad}\},$$

where we can verify that the first probability is essentially the probability of the sub-channel \mathbf{H} being bad and that the second one is essentially the probability of \mathbf{GH} being bad conditioned on the event that \mathbf{H} is not bad. As we know, the first probability decays with SNR as SNR^{-4} . To find out the SNR exponent of the second probability, we assume without loss of generality that the vector \mathbf{h}_1 is strong enough (since \mathbf{H} is not bad), as shown in Fig. 3(a). Now, we apply an orthogonal basis change from the canonical basis to $\{\mathbf{h}_1/\|\mathbf{h}_1\|, \mathbf{h}_1^\perp/\|\mathbf{h}_1^\perp\|\}$ and get the equivalent channel in Fig. 3(b). The basis change being an unitary transformation that is independent of the remaining parts of the channel, it does not affect the statistics of the rest of the channel. As shown in Fig. 3(b), the channel is bad if the three independent edges crossed by the “minimum cut” are bad. The probability for the latter to happen decays as SNR^{-3} , from which we conclude that the outage probability scales as $\text{SNR}^{-3} + \text{SNR}^{-4} \doteq \text{SNR}^{-3}$. Therefore, the mismatch between \mathbf{G} and \mathbf{H} is the dominating outage event and the end-to-end diversity of the $(2, 2, 2)$ channel with AF scheme is 3, as compared to 4 given by the cut-set bound.

IV. PARALLEL PARTITION

The naive AF scheme presented above can be seen as a space-only processing. In the point-to-point MIMO channel, it has been shown that space-only coding schemes (e.g., the V-BLAST scheme [31]) are suboptimal in diversity. Similarly, the AF scheme, as a space-only relaying scheme, does not achieve the maximum diversity in the multihop channel due to the mismatch between adjacent sub-channels. The clue is, just like the space-time codes achieve the maximum diversity in the point-to-point channel, space-time relay processing should be utilized in order to exploit the maximum distributed diversity in the multihop channel.

The first attempt was made in [20] with a distributed space-time coding scheme. In this scheme, each relay performs temporal random unitary transformation on the received signal from the source in an independent way. Then, they forward the transformed signal at the same time as if they were jointly sending a space-time codeword. The spatial correlation of the codewords is due to the fact that the received signal at different relays is from the same source. The temporal correlation, on the other hand, is brought in by the temporal transformation. In their setting where a single layer of relays and single-antenna terminals are assumed, the maximum diversity of the channel is achieved. This scheme is then generalized to the multi-antenna case [21] with structured algebraic transformations [32] instead of random transformations. However,

generalization of such schemes to the multihop case is difficult and the DMT is hard, if not impossible, to calculate.

In the following, we present a different approach to introduce the temporal processing. This approach does not depend on the dimension of the channel and thus suitable for multihop channels of arbitrary number of hops. The idea is to partition the relays in each layer. Based on the partition, the relays coordinately amplify-and-forward the received signal in a pre-assigned mode that changes periodically, which creates a parallel channel in the time domain. Such partition is thus called *parallel partition*. We show that the mismatch is removed in this way and the diversity upper bound is achieved.

In order to describe a parallel partition, some definitions and notations are needed. A *supernode* \mathcal{S} is a set of indices corresponding to a subset of antenna nodes in the same layer. The cardinality of \mathcal{S} is called the *size* of the supernode. An *edge* is defined as the channel between two antenna nodes from adjacent layers. An *AF path* is defined as a sequence of consecutive supernodes from the source to the destination, each supernode performing the AF operation. A *parallel partition* \mathcal{P} is defined as a set of AF paths. The number of AF paths in a partition is called the *partition size* and denoted by $|\mathcal{P}|$. An *independent parallel partition* is defined as a parallel partition where any two different AF paths do not share common edges. An independent partition of maximum size is called a *maximum partition*. An independent partition that achieves the maximum diversity d_{\max} is called a *full diversity partition*.

Lemma 4.1: For any fading channel defined by \mathbf{H} , we have

$$\mathbf{P} \{ \text{SNR} \|\mathbf{H}\|_{\text{F}}^2 < 1 \} \doteq \text{SNR}^{-d(0)}, \quad (24)$$

where $d(r)$ is the DMT of the channel.

Proof: See Appendix V-A. □

Lemma 4.2: Let us consider a set of K independent parallel AF channels

$$\mathbf{y}_k = \mathbf{\Pi}_k \mathbf{x}_k + \mathbf{z}_k, \quad k = 1, \dots, K,$$

where $\mathbf{\Pi}_k$'s are statistically independent. Then, the diversity order of the parallel channel is the sum of the diversity order of the individual AF sub-channels. Furthermore, if all the sub-channels have the same DMT $d_0(r)$, then the DMT of the parallel channel is $K d_0(r)$.

Proof: See Appendix V-B. □

A. Independent Parallel Partition

The independent parallel partition is accomplished in two steps : 1) partition each layer into supernodes, and 2) find K independent AF paths connecting the supernodes. Each AF path defines a relaying mode : only the supernodes in this path are on and perform the AF operation. Assume that a data frame of length KT is transmitted. Then, the relays change the relaying mode every T symbol times. We call it a *parallel AF scheme*, since the end-to-end channel is equivalent to a parallel AF channel in the time domain. Note that the AF scheme is the trivial partition of size 1 with a single “wide” AF path. As shown in remark 3.1, the trivial partition achieves the maximum diversity only when the wide AF path satisfies the conditions in (23). This being impossible in general, the parallel partition aims to find independent “narrow” paths each one of which satisfies the conditions in (23). And if the number of independent paths is large enough, then the maximum diversity order can be achieved according to lemma 4.2. Intuitively, the narrower the AF path is, the easier the conditions (23) are to be satisfied. In the extreme case with the narrowest AF path $(1, 1, \dots, 1)$, all conditions in (23) are met.

Lemma 4.3: In a (n_0, n_1, \dots, n_N) multihop channel, there are exactly d_{\max} independent single-antenna AF paths.

Proof: First, the converse is true, since otherwise, at least two AF paths share the same edge in the bottleneck of the channel. Then, the achievability is shown by construction : we connect the multihop channel in such a way that 1) there are d_{\max} incoming and outgoing edges for each intermediate layer, 2) the number of the incoming and outgoing edges is the same for each antenna node (say, in layer i) and can be either $\lfloor d_{\max}/n_i \rfloor$ or $\lceil d_{\max}/n_i \rceil$. This partition contains d_{\max} independent $(1, 1, \dots, 1)$ AF paths each one of which has diversity 1. \square

The lemma implies that the maximum partition is of size d_{\max} . From Lemma 4.2 and 4.3, the following proposition is immediate.

Proposition 4.1: With the parallel AF scheme, the DMT

$$d_{\max} (1 - r)^+ \quad (25)$$

is always achievable in a multihop channel of arbitrary number of hops and antennas.

Proof: The DMT (25) is simply achieved by applying the parallel AF scheme with the maximum partition. In this case, d_{\max} single-input-single-output (SISO) parallel sub-channels are generated, from which we have the DMT (25). \square

While the maximum diversity gain is achieved, this scheme only exploits one out of \tilde{n}_0 degrees of freedom of the channel. This is due to the SISO nature of the AF paths in the maximum partition. In order to improve the achievable multiplexing gain, we need parallel partitions with wider AF paths. Meanwhile, we still want the maximum diversity, which requires that the AF paths should not be too wide. The following theorem states a necessary and sufficient condition for an independent parallel partition to achieve the maximum diversity.

Theorem 4.1: Let the $n_{i^*} \times n_{i^*+1}$ channel be any bottleneck of the (n_0, n_1, \dots, n_N) multihop channel and \mathcal{P} be an independent partition of size K . Then, \mathcal{P} is a full diversity partition if and only if 1) $K = K_{i^*} K_{i^*+1}$ with $K_{i^*} \leq n_{i^*}$ and $K_{i^*+1} \leq n_{i^*+1}$, and 2) we have

$$\min_{i \notin \{i^*, i^*+1\}} n_{k,i} + 1 \geq n_{k,i^*} + n_{k,i^*+1}, \quad \forall k, \quad (26)$$

where $(n_{k,0}, \dots, n_{k,N})$ is the vector of numbers of antennas of the k th AF path.

Proof: To prove the theorem, let us assume there are respectively K_{i^*} and K_{i^*+1} supernodes in the layer i^* and layer i^*+1 , and define $K' \triangleq K_{i^*} K_{i^*+1}$. Then, we must have exactly $K (\leq K')$ connections between the supernodes from these two layers. The diversity of the partition \mathcal{P} is upper-bounded

$$d_{\mathcal{P}} \leq \sum_{k=1}^K n_{k,i^*} n_{k,i^*+1} \quad (27)$$

$$\leq \sum_{k=1}^{K'} n_{k,i^*} n_{k,i^*+1} \quad (28)$$

$$= n_{i^*} n_{i^*+1}.$$

Note that $d_{\max} = n_{i^*} n_{i^*+1}$ is achieved if and only if both (27) and (28) have equality. Thus, we must have both (26) according to the conditions in (23) and $K = K_{i^*} K_{i^*+1}$ at the same time. \square

Now, finding full diversity partitions with minimum size is an optimization problem that minimizes the partition size $|\mathcal{P}|$ subject to the constraint that \mathcal{P} must be an independent partition and satisfy the conditions given by theorem 4.1. Unfortunately, it remains an open problem for a general multihop channel. The main difficulty lies in the lack of knowledge on the mathematical structure of the independent partitions for a general multihop channel. Nevertheless, the problem is solved in the two-hop case.



Fig. 4. Two sets of parallel channels from the $(2, 2, 2)$ multihop channel.

Proposition 4.2: For a (n_0, n_1, n_2) channel, the minimum size of a full diversity partition is

$$K = \left\lceil \frac{n_1}{|n_0 - n_2| + 1} \right\rceil. \quad (29)$$

Proof: See Appendix VI-A. □

It is achieved by partitioning the relay layer into K supernodes of size $\lfloor \frac{n_1}{K} \rfloor$ or $\lceil \frac{n_1}{K} \rceil$. For example, the minimum partition size of the $(2, 4, 3)$ channel is 2 as compared to the maximum partition size 8; and each AF path is a $(2, 2, 3)$ channel instead of a $(1, 1, 1)$ channel. Another example is the (n, n, n) symmetric channel, where the minimum partition size is n as compared to the maximum partition size n^2 ; each AF path is a $(n, 1, n)$ channel.

Some words regarding the related previous works before proceeding further. In the relay channel with direct link and single layer of relays, the N -relay non-orthogonal AF (NAF) scheme [16] divides the data frame into N sub-frames, each one of which is relayed by one and only one relay. By creating a parallel NAF channel, this scheme is optimal in diversity. Similar thought was shown in [33] in the same channel setting with a different protocol called ND-RAF scheme. Removing the direct link from the channel setting, the scheme in [33] becomes the parallel AF scheme with the maximum partition in the single-antenna single-layer case.

B. Flip-and-Forward

With the parallel AF scheme, the maximum multiplexing gain of the channel is achieved only when every AF path in the partition achieves the maximum multiplexing gain $r_{\max} = \tilde{n}_0$. In the following, we propose a scheme that achieves both the maximum diversity gain and the maximum multiplexing gain. Let us consider an example first.

Example 4.1: The parallel channel $\{\mathbf{\Pi}_1, \mathbf{\Pi}_2\}$ in Fig. 4(a) has maximum diversity gain 4 and multiplexing gain 1, while the parallel channel $\{\mathbf{\Pi}'_1, \mathbf{\Pi}'_2\}$ in Fig. 4(b) has maximum diversity gain 4 and multiplexing gain 2.

In this example, $\{\mathbf{\Pi}_1, \mathbf{\Pi}_2\}$ corresponds to the parallel AF scheme based on the full diversity partition proposed by Proposition 4.2. However, it suffers from rate-deficiency, since both sub-channels are of rank 1. An alternative is the channel $\{\mathbf{\Pi}'_1, \mathbf{\Pi}'_2\}$ shown in Fig. 4(b). Note that

$$\begin{aligned}\mathbf{\Pi}_1 &= \mathbf{H}_2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{H}_1; & \mathbf{\Pi}_2 &= \mathbf{H}_2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{H}_1; \\ \mathbf{\Pi}'_1 &= \mathbf{H}_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{H}_1; & \mathbf{\Pi}'_2 &= \mathbf{H}_2 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{H}_1.\end{aligned}$$

Hence, we have

$$\begin{bmatrix} \mathbf{\Pi}'_1 & \mathbf{\Pi}'_2 \end{bmatrix} = \begin{bmatrix} \mathbf{\Pi}_1 & \mathbf{\Pi}_2 \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{I} & -\mathbf{I} \end{bmatrix}$$

from which $\|\mathbf{\Pi}'_1\|_F^2 + \|\mathbf{\Pi}'_2\|_F^2 = 2(\|\mathbf{\Pi}_1\|_F^2 + \|\mathbf{\Pi}_2\|_F^2)$. Therefore, according to lemma 4.1, they both achieve the maximum diversity gain 4 except that $\{\mathbf{\Pi}'_1, \mathbf{\Pi}'_2\}$ has the maximum multiplexing gain 2 as well. This scheme is called the *Amplify-Flip-and-Forward* (AFF)⁷ scheme, or simply the *Flip-and-Forward* (FF) scheme. The intuition behind the FF scheme is as follows. It has been shown that the mismatch between the two hops is the dominating outage event. Now, suppose that $\mathbf{\Pi}'_1$ is bad due to the bad “angle” between \mathbf{H}_1 and \mathbf{H}_2 both of which are not bad individually. Then, in the second sub-channel, an independent “rotation” matrix $\text{diag}\{1, -1\}$ is used to change the angle. With high probability, the new angle is not bad and the mismatch is solved.

In the light of the example, we generalize the scheme to arbitrary number of antennas and hops. Three steps are needed to describe the construction.

step 1 Find a full diversity independent parallel partition \mathcal{P} of size K . The partition defines the intermediate supernodes in each layer.

step 2 We denote the supernodes in layer i by $\mathcal{S}_{i,1}, \dots, \mathcal{S}_{i,K_i}$ with K_i being the number of supernodes in layer i . And we define the flip matrices $\mathbf{F}_{i,k}$ ’s as $n_i \times n_i$ diagonal matrices

⁷The processing matrices \mathbf{D}_i ’s have been neglected for simplicity of demonstration.

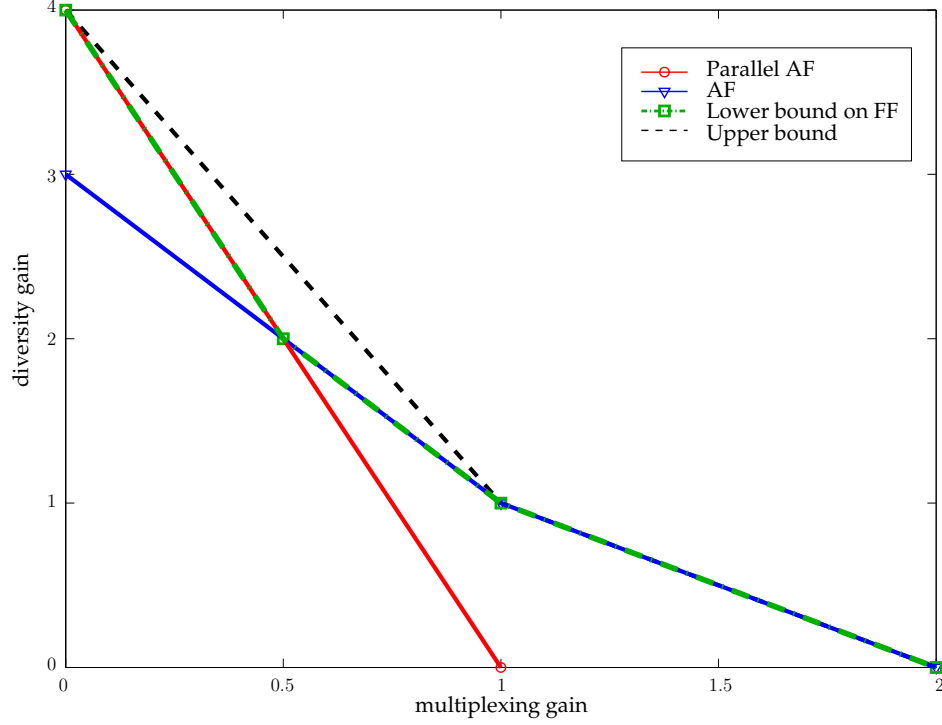


Fig. 5. Diversity-multiplexing tradeoff of $(2, 2, 2)$ channel with different schemes.

with

$$\mathbf{F}_{i,k}(j, j) = \begin{cases} -1, & \text{if } j \in \mathcal{S}_{i,k} \text{ and } k \neq 1, \\ 1, & \text{otherwise.} \end{cases}$$

step 3 The FF scheme is composed of $K' \triangleq \prod_{i=1}^{N-1} K_i$ parallel sub-channels $\{\Pi'_k\}_k$ with

$$\Pi'_k \triangleq \mathbf{H}_N \prod_{i=1}^{N-1} (\mathbf{F}_{i,f_i(k)} \mathbf{H}_i), \quad (30)$$

where $f_1(k) \triangleq (k-1)_{K_1} + 1$ and

$$f_i(k) \triangleq \left(\left\lceil \frac{k-1}{\prod_{j=1}^{i-1} K_j} \right\rceil \right)_{K_i} + 1, \quad i = 2, \dots, N-1.$$

In other words, the set of relays works in K' different flip modes, each one being identified by a sequence of flip modes of individual relay layers. And the mapping is effectuated by the functions $f_1(k), f_2(k), \dots, f_{N-1}(k)$. The exact DMT of the FF scheme being difficult to obtain, we get a lower bound instead.

Theorem 4.2: The FF scheme constructed above achieves the following DMT

$$d^{\text{FF}}(r) \geq d^{\text{AF}}(r) + (d_{\max} - d^{\text{AF}}(0)) (1 - K'r)^+, \quad \forall r. \quad (31)$$

Proof: See Appendix VI-B. \square

We can verify that $d^{\text{FF}}(0) = d_{\max}$, that is, the maximum diversity of the channel is achieved. Furthermore, the FF scheme is always better than the AF scheme, especially at low multiplexing gain. This can be explained by the intuition that the FF scheme solves the mismatch of adjacent hops using all possible combinations of flip modes of individual supernodes. The equivalent end-to-end channel of the FF scheme can be bad only if at least one of the hops are bad. The maximum diversity is thus achieved. Fig. 5 shows the DMT of different schemes in the channel of Example 4.1. While the AF and the parallel AF schemes achieve respectively the extreme of maximum multiplexing gain $(2, 0)$ and the extreme of maximum diversity gain $(0, 4)$, the FF scheme achieves both extremes.

Remark 4.1: The proposed FF scheme is constructed based on the flip matrices that are diagonal with ± 1 entries. In fact, it can be shown that a looser sufficient condition is for the matrices to be 1) diagonal, 2) linearly independent, and 3) of unit absolute value (power constraints). Therefore, we can find infinitely many sets of “flip” matrices that satisfy the above conditions and they are all diversity optimal. Intuitively, if the matrices are too “close”, the FF scheme tends to the AF scheme and the promised maximum diversity gain can be achieved only when the SNR is very large. This is translated into a poor power gain of the scheme. Hence, we should choose the matrices such that they are “far” from each other. In this way, with high enough probability, any mismatch can be solved by at least one “rotation” and the maximum diversity can be obtained in relatively small SNR. However, what remains open is how to choose the distance metric between the rotation matrices.

C. Non-Independent Partition

With independent partition, the total diversity is the sum of the diversity of each AF path. We also established some conditions that independent partitions must satisfy to achieve the maximum diversity. In the following, we investigate a particular case of non-independent partition.

Let us consider a parallel channel defined by $\{\Pi_k\}_k$ with

$$\Pi_k \triangleq \mathbf{H}_N \cdots \mathbf{H}_{i+1} \mathbf{J}_k \mathbf{H}_i \cdots \mathbf{H}_1, \quad (32)$$

where the *selection matrix* \mathbf{J}_k is a $n_i \times n_i$ diagonal matrix whose entries are zero except that $\mathbf{J}_k(k, k) = 1$. The matrices $\mathbf{\Pi}_k$'s are not independent, since they share the common sub-channels $\mathcal{G}1 \triangleq \mathbf{H}_{i-1} \cdots \mathbf{H}_1$ and $\mathcal{G}2 \triangleq \mathbf{H}_N \cdots \mathbf{H}_{i+2}$. However, the RP channels $\mathbf{H}_{i+1} \mathbf{J}_k \mathbf{H}_i$'s are independent for different k 's. Despite the dependency between the sub-channels, we can obtain the diversity order of the parallel channel.

Theorem 4.3: The diversity order of the channel described above is

$$\min\{d_{(n_0, \dots, n_i)}^{\text{AF}}(0), d_{(n_i, \dots, n_N)}^{\text{AF}}(0)\}. \quad (33)$$

Proof: We use the DMT interpretation given in Section III-B3 to sketch the proof. One possibility for the parallel channel $\{\mathbf{\Pi}_k\}_k$ to be in outage is that one of \mathcal{G}_1 and \mathcal{G}_2 is bad. The diversity is either $d_{(n_0, \dots, n_{i-1})}^{\text{AF}}(0)$ or $d_{(n_{i+1}, \dots, n_N)}^{\text{AF}}(0)$. Another possibility is that both \mathcal{G}_1 and \mathcal{G}_2 are good and that $\{\mathbf{\Pi}_k\}_k$ turns out to be bad. Without loss of generality, we assume that the flow from the source to layer $i - 1$ is k_1 and that from layer $i + 1$ to the destination is k_2 . And we call the outage event a *type*-(k_1, k_2) event. Then, it can be shown that $\{\mathbf{\Pi}_k\}_k$ is equivalent to $\{\mathbf{H}'_{i+1} \mathbf{J}_k \mathbf{H}'_i\}_k$ with $\mathbf{H}'_i \in \mathbb{C}^{n_i \times k_1}$ and $\mathbf{H}'_{i+1} \in \mathbb{C}^{k_2 \times n_i}$ being Gaussian matrices with i.i.d. $\mathcal{CN}(0, 1)$ entries. Now, we must disconnect all the sub-channels in $\{\mathbf{H}'_{i+1} \mathbf{J}_k \mathbf{H}'_i\}_k$, which costs $n_i \min\{k_1, k_2\}$. Therefore, the total cost for the type-(k_1, k_2) event is

$$d_{(n_0, \dots, n_{i-1})}^{\text{AF}}(k_1) + n_i \min\{k_1, k_2\} + d_{(n_{i+1}, \dots, n_N)}^{\text{AF}}(k_2).$$

The typical outage event is the one that minimizes the above total cost. For $k_2 \geq k_1$, using (20), we can show that the minimum total cost is $d_{(n_0, \dots, n_i)}^{\text{AF}}(0)$. Similarly, $d_{(n_{i+1}, \dots, n_N)}^{\text{AF}}(0)$ is the minimum total cost for $k_1 > k_2$. Since both costs are smaller than $d_{(n_0, \dots, n_{i-1})}^{\text{AF}}(0)$ and $d_{(n_{i+1}, \dots, n_N)}^{\text{AF}}(0)$ for the monotonicity (Corollary 3.2), we proved the theorem. \square

Note that with this particular partition at layer i , we achieve a diversity order as if layer i were clustered and the cooperative DF scheme were used. This result implies that one might achieve the maximum diversity with a partition of small size. For example, the maximum diversity order of the (3, 2, 2, 2, 3) channel is 4 and all the full diversity independent partitions are of size $K = 8$, i.e., eight (3, 1, 1, 1, 3) sub-channels. With the non-independent partition described above, we get a couple of (3, 2, 1, 2, 3) sub-channels, i.e., size 2. Since $d_{(2, 2, 3)}^{\text{AF}}(0) = 4$, the maximum diversity 4 is achieved as well according to Theorem 4.3.

We can apply the FF scheme to the case of non-independent partition. Then, in this example, the parallel channel is $\{\mathbf{\Pi}'_k\}_k$ with $\mathbf{\Pi}'_k \triangleq \mathbf{H}_N \cdots \mathbf{H}_{i+1} \mathbf{F}_k \mathbf{H}_i \cdots \mathbf{H}_1$ where the flip matrix \mathbf{F}_k is a

$n_i \times n_i$ diagonal matrix whose entries are one except that $\mathbf{F}_k(k, k) = -1$ if $k \neq 1$. The channels $\{\mathbf{\Pi}'_k\}_k$ being a linear invertible transformation of $\{\mathbf{\Pi}_k\}_k$, the generalized FF scheme achieves the diversity given by (33).

D. Extensions

With the nice parallel-channel structure, the FF scheme can be extended to various cases.

Let us first consider the extension to the MIMO relay channel with direct link and a single layer of N relays. By applying directly the single-antenna NAF scheme [16] to the multi-antenna case, the source cooperates with one relay at a time. This is equivalent to using the parallel AF scheme in the source-relays-destination link. The DMT lower bound is obtained in [34] as

$$d_{\mathbf{F}}(r) + N d_{(n_t, n, n_r)}^{\text{AF}}(2r), \quad (34)$$

where $d_{\mathbf{F}}(r)$ is the DMT of the $n_t \times n_r$ source-destination channel \mathbf{F} and each relay has n antennas. In fact, this lower bound can be improved to

$$d_{\mathbf{F}}(r) + d_{(n_t, N n, n_r)}^{\text{FF}}(2r), \quad (35)$$

by replacing the parallel AF scheme in the source-relays-destination link with the FF scheme. Comparing the second terms from (34) and (35), the gain in diversity of the new scheme over the MIMO NAF is reflected by

$$\begin{aligned} N d_{(n_t, n, n_r)}^{\text{AF}}(0) &\leq N n \min\{n_t, n_r\} \\ &= d_{(n_t, N n, n_r)}^{\text{FF}}(0) \end{aligned} \quad (36)$$

where the inequality (36) becomes strict when n is large. The gain in multiplexing of the source-relays-destination link is obvious when n is small, i.e., $n < \min\{n_t, n_r\}$. In this case, the FF scheme pools the relay antennas together to provide more degrees of freedom.

Another extension is to the multiuser case. Let us take the multiple access channel as an example. For simplicity, we assume that M users try to communicate with the common destination through the same layers of relays. Then, in the FF scheme, we have an equivalent parallel multiple access channel with

$$\mathbf{y}_k = \sum_{i=1}^M \mathbf{\Pi}_{k,i} \mathbf{x}_i + \mathbf{z}_k, \quad k = 1, \dots, K', \quad (37)$$

where $\{\Pi_{k,i}\}_k$ is similarly defined as in (30) with

$$\Pi_{i,k} \triangleq \mathbf{H}_N \mathbf{F}_{N-1, f_{N-1}(k)} \mathbf{H}_{N-1} \cdots \mathbf{H}_2 \mathbf{F}_{1, f_1(k)} \mathbf{H}_{1,k}. \quad (38)$$

Note that only the first hop is distinct for different users. Using the techniques of [24] and the our results for the single-user FF scheme, it is possible to analyze the DMT of the FF scheme in the multiple access channel. It is trivial to show that similar extension also holds for the broadcast channels with minor modifications.

V. THE CLUSTERED CASE REVISITED

In Section II-D, it has been shown that the cooperative DF scheme achieves the DMT cut-set bound in the clustered case. In this section, we would like to study some alternative schemes, since it might be impossible or unnecessary for all the clusters to decode the source message in some cases.

A. Serial Partition

The AF and the cooperative DF schemes can in fact be seen as two extremes of what we call the *serial partition* of multihop channels, defined as follows.

Definition 5.1: A *serial partition* is defined by a set of layer indices $\mathcal{D} \triangleq \{\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_{|\mathcal{D}|}\}$ with $0 < \mathcal{D}_1 < \mathcal{D}_2 < \dots < \mathcal{D}_{|\mathcal{D}|-1} < \mathcal{D}_{|\mathcal{D}|} \triangleq N$, each layer performing cooperative decoding-and-forward operation.

With a serial partition, the multihop channel becomes a serial concatenation of $|\mathcal{D}|$ AF channels. As in (4), the DMT of the multihop channel with any partition \mathcal{D} is easily derived as

$$d_{\mathcal{D}}(r) = \min_{i=1, \dots, |\mathcal{D}|} d_{(n_{\mathcal{D}_{i-1}}, \dots, n_{\mathcal{D}_i})}^{\text{AF}}(r), \quad (39)$$

where we defined $\mathcal{D}_0 \triangleq 0$. To get the maximum diversity gain, the question of *when to decode* has been answered earlier : when the conditions in (23) are not met. Another question is *where to decode*, i.e., how to find the partition of minimum size that achieves a given diversity order.

Proposition 5.1: Let us take $\mathcal{D}_0 = 0$ and we successively decide \mathcal{D}_i as the maximum integer in $(\mathcal{D}_{i-1}, N]$ such that

$$d_{(n_{\mathcal{D}_{i-1}}, \dots, n_{\mathcal{D}_i})}^{\text{AF}}(0) \geq d. \quad (40)$$

Then, the decoding set $\{\mathcal{D}_i\}$ defines the partition of minimum size that achieves a given diversity d ($\leq d_{\max}$).

Proof: From (39), it is easy to show that the proposed partition achieves diversity d . Now, we would like to show that the size of the proposed partition is minimized. To this end, it is enough to show that for any set \mathcal{D}' of decoding points that achieves diversity d , we have $\mathcal{D}'_i \leq \mathcal{D}_i$, $\forall i$. This is obviously true for \mathcal{D}'_1 , since the diversity of the AF channel degrades with the number of hops. By induction on i , it is shown that $\mathcal{D}'_{i+1} \leq \mathcal{D}_{i+1}$ because otherwise $(n_{\mathcal{D}_i}, \dots, n_{\mathcal{D}_{i+1}}) \subseteq (n_{\mathcal{D}'_i}, \dots, n_{\mathcal{D}'_{i+1}})$ and the corresponding diversity of the AF scheme cannot be larger than d according to the monotonicity of the DMT (Corollary 3.2). \square

The proposition matches the intuition that we should only decode when we have to, in the diversity sense. In other words, we allow for the degradation of diversity introduced by the AF operation, as long as the resulting diversity is larger than the target d .

B. CSI Aided Linear Processing

Another option is to linear process the received signal at each cluster without decoding it. Unlike the AF scheme in the non-clustered case, where trivial antenna-wise normalization is performed, we can run inter-antenna processing based on the available CSI at the cluster. With receiver CSI at the relays, let us consider the following project-and-forward (PF) scheme. At layer i , the received signal is first projected to the signal subspace spanned by the columns of the channel matrix \underline{H}_i . The dimension of the subspace is r_i , the rank of \underline{H}_i . After the component-wise normalization, the projected signal is transmitted using r_i (out of n_i) antennas. It is now clear that $\underline{H}_{i+1} \in \mathbb{C}^{n_{i+1} \times r_i}$ is actually composed of the r_i columns of the previously defined \underline{H}_{i+1} , with $r_0 \triangleq n_0$. More precisely, the $\underline{Q}_i \in \mathbb{C}^{n_i \times r_i}$ is an orthogonal basis with $\underline{Q}_i^\dagger \underline{Q}_i = \mathbf{I}$. We can rewrite

$$\underline{H}_i = \underline{Q}_i \underline{G}_i$$

with $\underline{G}_i \in \mathbb{C}^{r_i \times r_{i-1}}$. For simplicity, we let \underline{Q}_i be obtained by the QR decomposition [35] of \underline{H}_i if $n_i > r_{i-1}$ and be identity matrix if $n_i \leq r_i$. The spirit of the PF scheme is not to use more antennas than necessary to forward the signal. Since the useful signal lies only in the r_i -dimensional signal subspace, the projection of the received signal provides sufficient statistics and reduces the noise power by a factor $\frac{n_i}{r_i}$. In this case, only r_i antennas are needed to forward

the projected signal. Let us define $\mathbf{P}_i \triangleq \underline{\mathbf{D}}_i \mathbf{Q}_i^\dagger$. Then, as in the AF case, the PF multihop channel is equivalent to the channel defined by

$$\mathbf{\Pi}_{\text{PF}} = \underline{\mathbf{H}}_N \mathbf{P}_{N-1} \cdots \underline{\mathbf{H}}_2 \mathbf{P}_1 \underline{\mathbf{H}}_1.$$

The following proposition states that receiver CSI and inter-antenna processing do not improve the DMT of the AF scheme.

Proposition 5.2: The PF scheme is equivalent to the AF scheme.

Proof: See Appendix VI-C. □

While the PF and AF have the same DMT, the PF outperforms the AF in power gain for two reasons. One reason is, as stated before, that the projection reduces the average noise power. The other reason is that the accumulated noise in the AF case is more substantial than that in the PF case. This is because in the PF case, less relay antennas are used than in the AF case. Since the power of independent noises from different transmit antennas add up at the receiver side, the accumulated noise in the AF case “enjoys” a larger “transmit diversity order” than in the PF case.

On the other hand, if we could have receiver *and* transmitter CSI at the clusters, the DMT could be improved as shown by the following example.

Example 5.1: For a (n, n, \dots, n) clustered multihop channel, the DMT cut-set bound can be achieved by linear processing within clusters if both transmitter and receiver CSI are available at each cluster.

The optimum linear relaying scheme is defined by the processing matrices \mathbf{T}_i 's with $\mathbf{T}_i \triangleq \mathbf{V}_{i+1}^\dagger \mathbf{U}_i$ where we assume that $\mathbf{H}_i = \mathbf{U}_i^\dagger \mathbf{\Sigma}_i \mathbf{V}_i$ is the singular value decomposition of \mathbf{H}_i . The diagonal elements in the singular value matrix $\mathbf{\Sigma}_i$ are in increasing order. This scheme matches the adjacent hops by aligning the singular values in the same order. It is then equivalent to the channel defined by $\prod_i \mathbf{\Sigma}_i$, whose DMT can be shown⁸ to be as the $n \times n$ Rayleigh channel.

VI. CODES CONSTRUCTION

Now, we need codes that actually attain the DMT promised by the studied relaying strategies. To this end, the construction of Perfect STBCs [26], [27] for MIMO channels is extended to the multihop relay channels. The constructed codes are approximately universal [28].

⁸The proof, that is essentially as the proof in [23], is omitted here.

A. The Clustered Case

The relay clusters that perform the cooperative DF operation partition the multihop channel into a series of $|\mathcal{D}|$ MIMO channels, say, $\tilde{\mathbf{H}}_1, \tilde{\mathbf{H}}_2, \dots, \tilde{\mathbf{H}}_{|\mathcal{D}|}$ with $\tilde{\mathbf{H}}_i \in \mathbb{C}^{n_{\mathcal{D}_i} \times n_{\mathcal{D}_{i-1}}}$. An obvious coding scheme that achieves the DMT is described as follows. Let r be the target multiplexing gain. First, the source terminal encodes the message of $Tr \log \text{SNR}$ bits with a $n_0 \times T$ Perfect STBC $\mathcal{X}_0(r)$. Then, in a successive manner, layer \mathcal{D}_i tries to decode the message. When a success decoding is assumed, the $Tr \log \text{SNR}$ bits are encoded with a $n_{\mathcal{D}_i} \times T$ Perfect STBC $\mathcal{X}_i(r)$ and forwarded. We can show that as long as $T \geq T_{\min}$ with

$$T_{\min} \triangleq \max_{i=1, \dots, |\mathcal{D}|} n_{\mathcal{D}_{i-1}},$$

the series of Perfect STBCs $\{\mathcal{X}_i\}_i$ can be found [27]. With the union bound, the end-to-end error probability is upper-bounded

$$P_e(r, \text{SNR}) \leq \sum_{i=1}^{|\mathcal{D}|} P_e^{(i)}(r, \text{SNR}), \quad (41)$$

where $P_e^{(i)}$ is the error probability of $\mathcal{X}_i(r)$ in the MIMO sub-channel $\tilde{\mathbf{H}}_i$. Since $\mathcal{X}_i(r)$ is DMT-achieving for any fading statistics, we have

$$P_e^{(i)}(r, \text{SNR}) \doteq \text{SNR}^{-d_{(n_{\mathcal{D}_{i-1}}, \dots, n_{\mathcal{D}_i})}^{\text{AF}}(r)}. \quad (42)$$

From (41) and (42), the DMT (39) is achieved with coding delay T_{\min} . Since the Perfect STBCs are approximately universal [28], so is this coding scheme. Note that this scheme can be used for the AF and PF schemes with $|\mathcal{D}| = 1$.

B. The Non-Clustered Case

In the non-clustered case, the parallel AF and the FF schemes are used. Note that both schemes share the common parallel MIMO channel structure

$$\mathbf{y}_k = \mathbf{\Pi}_k \mathbf{x}_k + \mathbf{z}_k, \quad k = 1, \dots, K, \quad (43)$$

where $\mathbf{\Pi}_k \in \mathbb{C}^{n_{r,k} \times n_{t,k}}$ and K is the number of the parallel sub-channels. Let \mathcal{X} be a code for the parallel channel. A codeword is defined by a set of matrices $\{\mathbf{X}_k\}_{k=1}^K$ with $\mathbf{X}_k \in \mathbb{C}^{n_{t,k} \times T}$. We define a parallel STBC with non-vanishing determinant (NVD) as follows.

Definition 6.1: Let \mathcal{B} be an alphabet that is scalably dense, i.e., for $0 \leq a \leq 1$,

$$|\mathcal{B}(\text{SNR})| \doteq \text{SNR}^a, \quad \text{and} \\ s \in \mathcal{B}(\text{SNR}) \Rightarrow |s|^2 \leq \text{SNR}^a.$$

Then, a parallel STBC \mathcal{X} is called a *parallel NVD code* if it

- 1) is \mathcal{B} -linear⁹;
- 2) has full symbol rate, i.e., it transmits on average $\sum_k n_{t,k}$ symbols per channel use from the signal constellation \mathcal{B} ;
- 3) has the NVD property, i.e., for any pair of different codewords $\{\mathbf{X}_k\}_k, \{\hat{\mathbf{X}}_k\}_k \in \mathcal{X}$,

$$\prod_k \det \left((\mathbf{X}_k - \hat{\mathbf{X}}_k)(\mathbf{X}_k - \hat{\mathbf{X}}_k)^\dagger \right) \geq \kappa > 0, \quad (44)$$

with κ a constant independent of the SNR.

We have the following result.

Theorem 6.1: The parallel NVD codes are approximately universal over the parallel channel defined by (43).

Proof: See Appendix VI-D. □

Thus, to achieve the DMT of the parallel AF and the FF schemes, it is enough to construct a parallel NVD codes. Several remarks are made before proceeding to the code construction.

Remark 6.1: The actual data rate of the NVD codes is controlled by the size of the alphabet \mathcal{B} and the symbol rate. Efficient decoding schemes (e.g., sphere decoding) may not be implementable when the channel is under-determined or, alternatively speaking, rank-deficient in the sense that $\sum_k \text{rank}(\mathbf{\Pi}_k) < \sum_k n_{t,k}$. Practical schemes include reducing the symbol rate while increasing the size of the alphabet \mathcal{B} . This, however, does not guarantee the DMT-achievability.

Remark 6.2: Explicit parallel NVD codes for asymmetric parallel channel (i.e., $n_{t,i} \neq n_{t,j}$ for some $i \neq j$) being hard to construct algebraically, we focus on the symmetric case. Note that in the FF scheme, the equivalent parallel channel is always symmetric. In the parallel AF scheme, the numbers of transmit antennas of different sub-channels may be different. However, the problem can be overcome by using the same number of antennas (i.e., $\max_k n_{t,k}$). The resulting parallel channel has at least the same DMT as the original channel. Nevertheless, an

⁹ \mathcal{X} is \mathcal{B} -linear means that each entry of any codeword in \mathcal{X} is a linear combination of symbols from \mathcal{B} .

alternative code construction that is suitable for both symmetric and asymmetric parallel channels is provided in Appendix VI-E for completeness.

Remark 6.3: From a given parallel partition with size S , the number of the parallel sub-channels K is S in the parallel AF scheme, generally larger than S in the FF scheme. Since the minimum coding delay is $K \max_k n_{t,k}$ that grows linearly with K , it grows at least linearly with S . Moreover, the complexity of decoding can grow up to exponentially with K if ML decoding is used. That is why it is important to find partitions of small size S .

C. Algebraic Construction of Parallel NVD Codes

A systematic way to construct NVD codes is the construction from cyclic division algebra (CDA). For more details on the concept, the readers can refer to [36]. In the following, we aim to construct the Perfect symmetric parallel NVD codes with quadrature amplitude modulation (QAM) constellations.¹⁰ The generalization to hexagonal constellations is straightforward.

1) $K = 1$: We start by the construction of NVD codes for MIMO channels ($K = 1$). Let $\mathbb{L} \triangleq \mathbb{Q}(i, \theta)$ be a cyclic extension of degree n_t on the base field $\mathbb{Q}(i)$. We denote σ the generator of the Galois group $\text{Gal}(\mathbb{L}/\mathbb{Q}(i))$. Let $\gamma \in \mathbb{Q}(i)$ be such that $\gamma, \gamma^2, \dots, \gamma^{n_t-1}$ are non-norm elements in \mathbb{L} . Then, we can construct a CDA $\mathcal{A} = (\mathbb{L}/\mathbb{Q}(i), \sigma, \gamma)$ of degree n_t . Each element in \mathcal{A} has the following matrix representation

$$\Xi = \begin{pmatrix} x_0 & x_1 & \dots & x_{n_t-1} \\ \gamma\sigma(x_{n_t-1}) & \sigma(x_0) & \dots & \sigma(x_{n_t-2}) \\ \vdots & \vdots & \ddots & \vdots \\ \gamma\sigma^{n_t-1}(x_1) & \gamma\sigma^{n_t-1}(x_2) & \dots & \sigma^{n_t-1}(x_0) \end{pmatrix}, \quad (45)$$

where $x_i \in \mathcal{O}_{\mathbb{L}}, \forall i$. Since \mathcal{A} is a CDA, we can show that $\det \Xi \in \mathbb{Z}[i]$ and that the determinant is zero only when Ξ is a zero matrix. Thus, the NVD property is proved by considering that the difference matrix of each pair of codewords is in the form of Ξ .

It is usually desirable to get a STBC with good shaping. To this end, we can impose the additional constraint that the vectorized codeword is a rotated version of a QAM ^{$N n_t^2$} constellation, as known as the cubic constellation. Rotated constellations constructions from algebraic number fields are well-known now (see, e.g., [38] for a comprehensive tutorial on this topic). This can

¹⁰The construction was first reported in [37] and is included for sake of completeness.

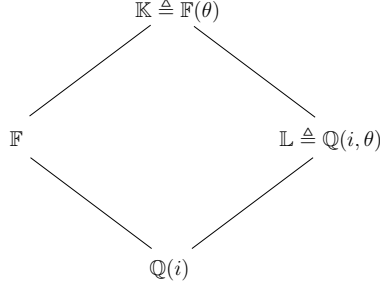


Fig. 6. Field extension tower.

be made possible if 1) x_i 's in the matrix Ξ belong to some properly chosen ideal $\mathcal{I} \subseteq \mathcal{O}_{\mathbb{L}}$ [39], and 2) $|\gamma| = 1$ (see [27] for a general method). The thus-constructed NVD codes are well-known as the Perfect STBCs.

2) $K > 1$: The construction of parallel NVD codes is similar to the construction presented above. First, we construct a CDA in the same manner as the previous case by simply 1) replacing the base field $\mathbb{Q}(i)$ by a new field \mathbb{F} , a Galois extension of degree K over $\mathbb{Q}(i)$; 2) replacing the field \mathbb{L} by $\mathbb{K} \triangleq \mathbb{F}(\theta)$, a cyclic extension of degree n_t over \mathbb{F} (same θ as the previous case); and 3) choosing γ such that $\gamma, \gamma^2, \dots, \gamma^{n_t-1}$ are non-norm elements in \mathbb{K} . We impose that $\mathbb{F} \cap \mathbb{L} = \mathbb{Q}(i)$. Note that the extension \mathbb{K}/\mathbb{F} remains cyclic with the same Galois group as $\text{Gal}(\mathbb{L}/\mathbb{Q}(i))$ (Fig. 6). Thus, the constructed CDA is $\mathcal{A}(\mathbb{K}/\mathbb{F}, \sigma, \gamma)$. Now, let $\{\tau_1, \tau_2, \dots, \tau_K\}$ be the Galois group of the extension $\mathbb{F}/\mathbb{Q}(i)$ and define

$$\Xi_k \triangleq \tau_k(\Xi), \quad k = 1, \dots, K,$$

where Ξ is the matrix representation of some element in \mathcal{A} and is in the form (45). Now, we have

$$\begin{aligned} \prod_k \det \Xi_k &= \prod_k \tau_k(\det \Xi) \\ &= N_{\mathbb{F}/\mathbb{Q}(i)}(\det \Xi) \end{aligned}$$

that is in $\mathbb{Z}[i]$. Finally, we construct codewords $\{\mathbf{X}_k\}_k$ in the form of $\{\Xi_k\}_k$ with QAM symbols and we can show that the difference matrix of a pair of different codewords is also in the form of $\{\Xi_k\}_k$ with symbols in $\mathbb{Z}[i]$. The NVD condition (44) is thus met. Similarly, the cubic

shaping can be obtained with the same kind of conditions mentioned before. An explicit code construction is provided in the following example.

Example 6.1 (Two transmit antennas and $K = 2^m$ sub-channels): Let us define $\zeta_{2^{m+1}} \triangleq e^{-i\pi 2^m}$. Then, we consider the base field $\mathbb{F} = \mathbb{Q}(\zeta_{2^{m+2}})$, an extension of $\mathbb{Q}(i)$ of degree 2^m and take $\mathbb{K} = \mathbb{F}(\sqrt{5}) = \mathbb{Q}(\zeta_{2^{m+2}}, \sqrt{5})$. We can verify that $\gamma \triangleq \zeta_{2^{m+2}}$ is a non-norm element in \mathbb{K} (see Appendix VI-F). Let $\theta = \frac{1+\sqrt{5}}{2}$ and $\sigma : \theta \mapsto \bar{\theta} = \frac{1-\sqrt{5}}{2}$. The ring of integers of \mathbb{K} is $\mathcal{O}_{\mathbb{K}} = \{a + b\theta \mid a, b \in \mathbb{Z}[\zeta_{2^{m+2}}]\}$. And the chosen ideal is principle, i.e., $\mathcal{I} = (\alpha)\mathcal{O}_{\mathbb{K}}$ with $\alpha = 1 + i - i\theta$. The matrix Ξ is given by

$$\Xi = \begin{bmatrix} \alpha \cdot (a + b\theta) & \alpha \cdot (c + d\theta) \\ \gamma\bar{\alpha} \cdot (c + d\bar{\theta}) & \bar{\alpha} \cdot (a + b\bar{\theta}) \end{bmatrix}, \quad (46)$$

where $a, b, c, d \in \mathbb{Z}[\zeta_{2^{m+2}}]$. We can show that the shaping property is satisfied and finally, this code is a perfect STBC for the parallel channel.

VII. NUMERICAL EXAMPLES

In this section, we present the numerical results on the proposed schemes. The performance measures are either the outage probability or the symbol error rate probability versus the average received SNR per bit. The results are obtained with Monte-Carlo simulations.

The first example is to illustrate the impact of vertical reduction of multihop channels, as shown in Fig. 8(a). In a $(1, 4, 1)$ channel, the necessary antenna number \bar{n} from (18) is 1 and the minimal vertical form is thus $(1, 1, 1)$. We observe that, with the same diversity order 1, an asymptotic power gain of 7 dB is obtained by using only one relay antennas out of four, if the AF scheme is used. The gain is due to the fact that using more relaying antennas hardens of relayed noise. In the $(3, 1, 4, 2)$ channel, the necessary number of antennas \bar{n} from (18) is 2. As shown in Fig. 8(a), by restricting the number of relay antennas to 2, we have a $(3, 1, 2, 2)$ channel and an asymptotic power gain of 2 dB is observed. We can further reduce the number of transmit antennas to 2 to get a $(2, 1, 2, 2)$ channel. Unlike the reduction of relay antennas, the reduction of transmit antennas does not provide any gain because it does not impact the relayed noise. In contrast, it degrades the performance since the first hop $(2, 1)$ is faded more seriously than the original first hop $(3, 1)$. Nevertheless, the $(2, 1, 2, 2)$ channel is still better than the $(3, 1, 4, 2)$ channel and is only 0.7 dB from the $(3, 1, 2, 2)$ channel. The coded performance of the $(3, 1, 4, 2)$

channel is then studied Fig. 8(b). The diagonal algebraic space-time (DAST) code¹¹ [40] can be used. As shown in Fig. 8(b), with the DAST code, the symbol error rate performances of in the $(3, 1, 4, 2)$, $(3, 1, 2, 2)$ and $(2, 1, 2, 2)$ channels have exactly the same behavior as the outage performances of the channels do Fig. 8(a). Moreover, the reduction in the number of transmit antenna allows us to use the Alamouti code [41] (the $(2, 1, 2, 2)$ channel). As we can see in the figure, the Alamouti code, besides the advantage of lower decoding complexity, outperforms all the DAST codes. The potential benefits from the vertical reduction are thus highlighted.

Then, we consider the parallel partition of two multihop channels : the $(2, 2, 2, 2)$ and $(2, 4, 3)$ channels. The resulting AFF scheme is compared to the AF scheme in terms of both the outage probability and the symbol error rate. With the AFF scheme, we create respectively four and two parallel sub-channels with two transmit antennas for the $(2, 2, 2, 2)$ and $(2, 4, 3)$ channels. Specifically, the AFF scheme for the $(2, 2, 2, 2)$ channel is based on a partition of four $(2, 1, 1, 2)$ sub-channels and for the $(2, 4, 3)$ channel is a partition of two $(2, 2, 3)$ sub-channels. As shown in Fig. 9(a), the diversity order of the AFF scheme for the $(2, 2, 2, 2)$ (respectively, $(2, 4, 3)$ channel) is 4 (respectively, 8), as compared to that of the AF scheme (3 and 6, respectively). The coded performance is also studied. We apply the construction provided by Example 6.1 to get Perfect parallel STBCs for two and four sub-channels. As we can observe in Fig. 9(b), with the use of Perfect codes, the symbol error rate performance has similar behaviors as the outage performance.

The last example is a $(3, 1, 4, 2)$ channel in the clustered case. Through this example, we would like to address the impact of “where to decode” on the end-to-end performance. The all-AF and all-DF schemes correspond respectively to the case with no decoding relay cluster and that with two decoding relay clusters. With one decoding cluster, the choice is made between decoding at the first cluster and decoding at the second one. As shown in Fig. 10, the all-AF scheme has diversity order two and the all-DF scheme has diversity order 3 as analytically expected. With only one decoding cluster, the diversity order is also predictable : diversity two in the single-antenna cluster and diversity 3 in the four-antenna cluster. What is impressive in this example is that the two curves with different choices of decoding cluster joins the all-AF and all-DF curves respectively at high SNR. Therefore, only one decoding cluster is enough to achieve

¹¹Note that the DAST code is the diagonal version of the rate-one Perfect code proposed in [26].

good performance in this case. And the decoding cluster should not be the single-antenna node.

VIII. CONCLUSION

The diversity of MIMO multihop relay channels has been investigated in both the clustered and non-clustered cases. Our results showed that, in both cases, the maximum diversity gain and the maximum multiplexing gain of the multihop channel can be achieved. In the clustered case, the optimal scheme is cooperative decode-and-forward that achieves the upper bound on the diversity-multiplexing tradeoff of the channel. In the non-clustered case, the key to achieve the maximum diversity is space-time relay processing. Our approach is to introduce temporal processing to the amplify-and-forward scheme by creating a parallel channel in the time domain. We proposed a flip-and-forward that achieves both the maximum diversity and multiplexing gain of an arbitrary multihop channel in a distributed manner. We also showed that the FF scheme can be easily extended to the multiuser case. With its low relaying and signaling complexity, the FF scheme is suitable for wireless *ad hoc* networks. Approximately universal coding schemes have been proposed for all the relaying strategies studied in this work.

APPENDIX I

PRELIMINARIES

The followings are some preliminary results that are essential to the proofs.

Lemma A1.1 (Calculation of diversity-multiplexing tradeoff): Consider a linear fading Gaussian channel defined by \mathbf{H} for which $\det(\mathbf{I} + \text{SNR} \mathbf{H} \mathbf{H}^\dagger)$ is a function of $\boldsymbol{\lambda}$, a vector of positive random variables. Then, the DMT $d_{\mathbf{H}}(r)$ of this channel can be calculated as

$$d_{\mathbf{H}}(r) = \inf_{\mathcal{O}(\boldsymbol{\alpha}, r)} E(\boldsymbol{\alpha})$$

where $\alpha_i \triangleq -\log v_i / \log \text{SNR}$ is the exponent of v_i , $\mathcal{O}(\boldsymbol{\alpha}, r)$ is the outage event set in terms of $\boldsymbol{\alpha}$ and r in the high SNR regime, and $E(\boldsymbol{\alpha})$ is the exponential order of the pdf $p(\boldsymbol{\alpha})$, i.e.,

$$p(\boldsymbol{\alpha}) \doteq \text{SNR}^{-E(\boldsymbol{\alpha})}.$$

Proof: This lemma can be justified by (2) using Laplace's method, as shown in [23]. \square

Definition A1.1 (Wishart Matrix): The $m \times m$ random matrix $\mathbf{W} = \mathbf{H} \mathbf{H}^\dagger$ is a (central) complex Wishart matrix with n degrees of freedom and covariance matrix \mathbf{R} (denoted as $\mathbf{W} \sim \mathcal{W}_m(n, \mathbf{R})$), if the columns of the $m \times n$ matrix \mathbf{H} are zero-mean independent complex Gaussian vectors with covariance matrix \mathbf{R} .

Lemma A1.2: The joint pdf of the eigenvalues of $\mathbf{W} \triangleq \mathbf{H} \mathbf{H}^\dagger \sim \mathcal{W}_m(n, \mathbf{R}_{m \times m})$ is identical to that of any $\mathbf{W}' \sim \mathcal{W}_{m'}(n, \text{diag}(\mu_1, \dots, \mu_{m'}))$ if $\mu_1 \geq \dots \geq \mu_{m'} > \mu_{m'+1} = \dots = \mu_m = 0$ are the eigenvalues of $\mathbf{R}_{m \times m}$.

Proof: Apply the eigenvalue decomposition on \mathbf{R} and the result is immediate using the unitary invariance property [42] of Wishart matrices. \square

Lemma A1.3 ([43]–[46]): Let \mathbf{W} be a central complex Wishart matrix $\mathbf{W} \sim \mathcal{W}_m(n, \mathbf{R})$, where the eigenvalues of \mathbf{R} are distinct¹² and their ordered values are $\mu_1 > \dots > \mu_m > 0$. Let $\lambda_1 > \dots > \lambda_q > 0$ be the ordered positive eigenvalues of \mathbf{W} with $q \triangleq \min\{m, n\}$. The joint

¹²In the particular case where some eigenvalues of \mathbf{R} are identical, we apply the l'Hospital rule to the pdf obtained, as shown in [45].

pdf of λ conditioned on μ is

$$p(\lambda|\mu) = \begin{cases} K_{m,n} \text{Det}(\Omega_1) \prod_{i=1}^m \mu_i^{m-n-1} \lambda_i^{n-m} \prod_{i<j}^m \frac{\lambda_i - \lambda_j}{\mu_i - \mu_j}, & \text{if } n \geq m, \end{cases} \quad (47a)$$

$$p(\lambda|\mu) = \begin{cases} G_{m,n} \text{Det}(\Omega_2) \prod_{i<j}^m \frac{1}{(\mu_i - \mu_j)} \prod_{i<j}^n (\lambda_i - \lambda_j), & \text{if } n < m, \end{cases} \quad (47b)$$

where $K_{m,n}$ and $G_{m,n}$ are normalization factors; $\text{Det}(\cdot)$ denotes the absolute value of the determinant $\det(\cdot)$; $\Omega_1 \triangleq [e^{-\lambda_j/\mu_i}]_{i,j=1}^m$ and

$$\Omega_2 \triangleq \begin{bmatrix} 1 & \mu_1 & \cdots & \mu_1^{m-n-1} & \mu_1^{m-n-1} e^{-\frac{\lambda_1}{\mu_1}} & \cdots & \mu_1^{m-n-1} e^{-\frac{\lambda_n}{\mu_1}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & \mu_m & \cdots & \mu_m^{m-n-1} & \mu_m^{m-n-1} e^{-\frac{\lambda_1}{\mu_m}} & \cdots & \mu_m^{m-n-1} e^{-\frac{\lambda_n}{\mu_m}} \end{bmatrix}. \quad (48)$$

In the non-correlated case with $\mathbf{R} = \mathbf{I}$, the joint pdf is

$$P_{m,n} e^{-\sum_i \lambda_i} \prod_{i=1}^q \lambda_i^{|m-n|} \prod_{i<j}^q (\lambda_i - \lambda_j)^2. \quad (49)$$

Now, let us define the *eigen-exponents*

$$\alpha_i \triangleq -\log \lambda_i / \log \text{SNR}, \quad i = 1, \dots, q, \quad \text{and} \quad \beta_i \triangleq -\log \mu_i / \log \text{SNR}, \quad i = 1, \dots, m.$$

Lemma A1.4:

$$\text{Det}(\Omega_1) \doteq \begin{cases} \text{SNR}^{-E_{\Omega_1}(\alpha, \beta)}, & \text{for } (\alpha, \beta) \in \mathcal{R}^{(1)} \\ \text{SNR}^{-\infty}, & \text{otherwise,} \end{cases} \quad (50)$$

where

$$E_{\Omega_1}(\alpha, \beta) \triangleq \sum_{j=1}^m \sum_{i<j} (\alpha_i - \beta_j)^+, \quad (51)$$

and

$$\mathcal{R}^{(1)} \triangleq \{\alpha_1 \leq \dots \leq \alpha_m, \beta_1 \leq \dots \leq \beta_m, \text{ and } \beta_i \leq \alpha_i, \text{ for } i = 1, \dots, m\}. \quad (52)$$

Proof: Please refer to [34] for details. \square

Lemma A1.5:

$$\text{Det}(\Omega_2) \doteq \begin{cases} \text{SNR}^{-E_{\Omega_2}(\alpha, \beta)}, & \text{for } (\alpha, \beta) \in \mathcal{R}^{(2)} \\ \text{SNR}^{-\infty}, & \text{otherwise,} \end{cases} \quad (53)$$

where

$$E_{\Omega_2}(\alpha, \beta) \triangleq \sum_{i=1}^n (m-n-1)\beta_i + \sum_{i=n+1}^m (m-i)\beta_i + \sum_{j=1}^n \sum_{i<j} (\alpha_i - \beta_j)^+ + \sum_{j=n+1}^m \sum_{i=1}^n (\alpha_i - \beta_j)^+ \quad (54)$$

and

$$\mathcal{R}^{(2)} \triangleq \{\alpha_1 \leq \dots \leq \alpha_n, \beta_1 \leq \dots \leq \beta_m, \text{ and } \beta_i \leq \alpha_i, \text{ for } i = 1, \dots, n\}. \quad (55)$$

Proof: First, we have

$$\text{Det}(\Omega_2) = \prod_{i=1}^m \mu_i^{m-n-1} \text{Det} \begin{bmatrix} \mu_1^{-(m-n-1)} & \dots & 1 & e^{-\lambda_1/\mu_1} & \dots & e^{-\lambda_n/\mu_1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \mu_m^{-(m-n-1)} & \dots & 1 & e^{-\lambda_1/\mu_m} & \dots & e^{-\lambda_n/\mu_m} \end{bmatrix}. \quad (56)$$

Then, let us denote the determinant in the RHS of (56) as D and we rewrite it as

$$D = \text{Det} \begin{bmatrix} d_{1,m}^{(m-n-1)} & \dots & 0 & e^{-\lambda_1/\mu_1} - e^{-\lambda_1/\mu_m} & \dots & e^{-\lambda_n/\mu_1} - e^{-\lambda_n/\mu_m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ d_{m-1,m}^{(m-n-1)} & \dots & 0 & e^{-\lambda_1/\mu_{m-1}} - e^{-\lambda_1/\mu_m} & \dots & e^{-\lambda_n/\mu_{m-1}} - e^{-\lambda_n/\mu_m} \\ \mu_m^{-(m-n-1)} & \dots & 1 & e^{-\lambda_1/\mu_m} & \dots & e^{-\lambda_n/\mu_m} \end{bmatrix} \quad (57)$$

$$\doteq \text{Det} \begin{bmatrix} d_{1,m}^{(m-n-1)} & \dots & d_{1,m}^{(1)} & e^{-\lambda_1/\mu_1} & \dots & e^{-\lambda_n/\mu_1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ d_{m-1,m}^{(m-n-1)} & \dots & d_{m-1,m}^{(1)} & e^{-\lambda_1/\mu_{m-1}} & \dots & e^{-\lambda_n/\mu_{m-1}} \end{bmatrix} \prod_{i=1}^n (1 - e^{-\lambda_i/\mu_m}) \quad (58)$$

where $d_{i,j}^{(k)} \triangleq \mu_i^{-k} - \mu_j^{-k}$ and the product term in (58) is obtained since $1 - e^{-(\lambda_i/\mu_m - \lambda_i/\mu_j)} \doteq 1 - e^{-\lambda_i/\mu_m}$ for all $j < m$. Let us denote the determinant in (58) as D_m . Then, by multiplying the first column in D_m with μ_m^{m-n-1} and noting that $\mu_m^{m-n-1} d_{i,m}^{(m-n-1)} = 1 - (\mu_m/\mu_i)^{m-n-1} \approx 1$, the first column of D_m becomes all 1. Now, by eliminating the first $m-2$ “1”s of the first column by subtracting all rows by the last row as in (57) and (58), we have $\mu_m^{m-n-1} D_m \doteq \prod_{i=1}^n (1 - e^{-\lambda_i/\mu_m}) D_{m-1}$. By continuing reducing the dimension, we get

$$\begin{aligned} \text{Det}(\Omega_2) &\doteq \text{Det} [e^{-\lambda_j/\mu_i}]_{i,j=1}^n \prod_{i=1}^{n+1} \mu_i^{m-n-1} \prod_{i=n+2}^m \mu_i^{m-i} \\ &\quad \cdot \prod_{i=1}^n \prod_{j=n+1}^m (1 - e^{-\lambda_i/\mu_j}) \end{aligned}$$

from which we prove the lemma, by applying (50). \square

With the preceding lemmas, we have the following lemma that provides the asymptotical pdf of α conditioned on β in the high SNR regime.

Lemma A1.6:

$$p(\alpha|\beta) \doteq \begin{cases} \text{SNR}^{-E(\alpha|\beta)}, & \text{for } (\alpha, \beta) \in \mathcal{R}_{\alpha|\beta}, \\ \text{SNR}^{-\infty}, & \text{otherwise,} \end{cases} \quad (59)$$

where

$$E(\boldsymbol{\alpha}|\boldsymbol{\beta}) \triangleq \sum_{i=1}^q (n+1-i)\alpha_i + \sum_{i=1}^q (i-n-1)\beta_i + \sum_{j=1}^q \sum_{i < j} (\alpha_i - \beta_j)^+ + \sum_{j=q+1}^m \sum_{i=1}^q (\alpha_i - \beta_j)^+, \quad (60)$$

and

$$\mathcal{R}_{\boldsymbol{\alpha}|\boldsymbol{\beta}} \triangleq \{\alpha_1 \leq \dots \leq \alpha_q, \beta_1 \leq \dots \leq \beta_m, \text{ and } \beta_i \leq \alpha_i, \text{ for } i = 1, \dots, q\}. \quad (61)$$

Proof: Let us replace $\text{Det}(\boldsymbol{\Omega}_1)$ and $\text{Det}(\boldsymbol{\Omega}_2)$ in (47a) and (47b) using the results of Lemma A1.4 and Lemma A1.5. Then, by applying variable changes as done in [23], (60) can be obtained after some elementary manipulations. \square

When $\mathbf{R} = \mathbf{I}$, i.e., $\mu_1 = \dots = \mu_m = 1$, the joint pdf of $\boldsymbol{\alpha}$ is found in [23] as shown in the following lemma.

Lemma A1.7:

$$p(\boldsymbol{\alpha}) \doteq \begin{cases} \text{SNR}^{-\sum_{i=1}^q (m+n+1-2i)\alpha_i}, & \text{for } \boldsymbol{\alpha} \in \mathcal{R}_{\boldsymbol{\alpha}}, \\ \text{SNR}^{-\infty}, & \text{otherwise,} \end{cases} \quad (62)$$

with $\mathcal{R}_{\boldsymbol{\alpha}} \triangleq \{0 \leq \alpha_1 \leq \dots \leq \alpha_q\}$.

This lemma can be justified either by using (49) or by setting $\beta_i = 0$, $\forall i$ in (60).

Lemma A1.8 ([47]): Let \mathbf{M} be any $m \times n$ random matrix and \mathbf{T} be any $m \times m$ non-singular matrix whose singular values satisfy $\sigma_{\min}(\mathbf{T}) \doteq \sigma_{\max}(\mathbf{T}) \doteq \text{SNR}^0$. Define $q \triangleq \min\{m, n\}$ and $\tilde{\mathbf{M}} \triangleq \mathbf{T}\mathbf{M}$. Let $\sigma_1(\mathbf{M}) \geq \dots \geq \sigma_q(\mathbf{M})$ and $\sigma_1(\tilde{\mathbf{M}}) \geq \dots \geq \sigma_q(\tilde{\mathbf{M}})$ be the ordered singular values of \mathbf{M} and $\tilde{\mathbf{M}}$. Then, we have

$$\sigma_i(\tilde{\mathbf{M}}) \doteq \sigma_i(\mathbf{M}), \quad \forall i.$$

APPENDIX II

PROOF OF THEOREM 3.1

Proposition A2.1: Let us denote the non-zero ordered eigenvalues of $\mathbf{\Pi}\mathbf{\Pi}^\dagger$ by $\lambda_1 \geq \dots \geq \lambda_{n_{\min}} > 0$ with $n_{\min} \triangleq \min_{i=0, \dots, N} n_i$. Then, the joint pdf of the eigen-exponents $\boldsymbol{\alpha}$ satisfies

$$p(\boldsymbol{\alpha}) \doteq \begin{cases} \text{SNR}^{-E(\boldsymbol{\alpha})}, & \text{for } 0 \leq \alpha_1 \leq \dots \leq \alpha_{n_{\min}}, \\ \text{SNR}^{-\infty}, & \text{otherwise,} \end{cases} \quad (63)$$

where

$$E(\boldsymbol{\alpha}) \triangleq \sum_{i=1}^{n_{\min}} c_i \alpha_i \quad (64)$$

with c_i 's defined by (14).

From Lemma A1.1, we can derive the DMT with the following optimization problem

$$d(r) = \min_{\alpha \in \mathcal{O}_0(r)} \sum_i c_i \alpha_i$$

with $\mathcal{O}_0(r) \triangleq \{\sum_i (1 - \alpha_i)^+ \leq r\}$ being the outage region. Note that c_i is decreasing and α_i is increasing with respect to i . Then, the proof of Theorem 3.1 is immediate.

Now, what remains is the proof of Proposition A2.1. The following lemma will be needed in the proof.

Lemma A2.1: Let $\mathcal{I}_k \triangleq [p_k, p_{k-1}]$, $k = 1, \dots, N$, be N consecutively joint intervals with $p_N \triangleq -\infty$, $p_0 \triangleq \tilde{n}_0$, and

$$p_k \triangleq \sum_{l=0}^k \tilde{n}_l - k\tilde{n}_{k+1} \quad k = 1, \dots, N-1. \quad (65)$$

Then, we have

$$c_i = 1 - i + \left\lfloor \frac{\sum_{l=0}^k \tilde{n}_l - i}{k} \right\rfloor, \quad \text{for } i \in \mathcal{I}_k. \quad (66)$$

Proof: c_i defined by (14) is the minimum of N sequences corresponding to the N values of k . It is enough to show that each of the N sequences dominates in a consecutive manner. We omit the details here. \square

A. Sketch of the Proof of Proposition A2.1

The proof will be by induction on N . From lemma A1.7, the proposition is trivial for $N = 1$. Suppose the proposition holds for some N and $\mathbf{\Pi} \triangleq \mathbf{H}_1 \cdots \mathbf{H}_N$, we would like to show that it is also true for $N + 1$ and $\mathbf{\Pi}' \triangleq \mathbf{H}_1 \cdots \mathbf{H}_{N+1}$. For simplicity, the “primed” notations (e.g., α' , \mathbf{n}' , $\tilde{\mathbf{n}}'$, \mathbf{c}' , n'_{\min} , etc.) will be used for the respective parameters of $\mathbf{\Pi}'$. Note that $\mathbf{\Pi}'(\mathbf{\Pi}')^\dagger \sim \mathcal{W}_{n_0}(n_{N+1}, \mathbf{\Pi}\mathbf{\Pi}^\dagger)$ for a given $\mathbf{\Pi}$, since $\mathbf{\Pi}' = \mathbf{\Pi}\mathbf{H}_{N+1}$. According to lemma A1.2, the pdf of the eigenvalues λ' of $\mathbf{\Pi}'(\mathbf{\Pi}')^\dagger$ is identical to that of $\mathcal{W}_{n_{\min}}(n_{N+1}, \text{diag}(\lambda))$. Hence, the pdf of α' can

be obtained as the marginal pdf of (α', α)

$$\begin{aligned} p(\alpha') &= \int_{\mathbb{R}^{n_{\min}}} p(\alpha', \alpha) d\alpha \\ &= \int_{\mathbb{R}^{n_{\min}}} p(\alpha' | \alpha) p(\alpha) d\alpha \\ &\doteq \int_{\mathcal{R}} \text{SNR}^{-E(\alpha' | \alpha)} \text{SNR}^{-E(\alpha)} d\alpha \end{aligned} \quad (67)$$

$$\doteq \text{SNR}^{-\hat{E}(\alpha')}, \quad (68)$$

where (67) comes from lemma A1.6 and our assumption that (63) holds for N , with

$$\begin{aligned} \mathcal{R} &\triangleq \mathcal{R}_{\alpha' | \alpha} \cap \mathcal{R}_{\alpha} \\ &= \left\{ 0 \leq \alpha'_1 \leq \dots \leq \alpha'_{n'_{\min}}, \ 0 \leq \alpha_1 \leq \dots \leq \alpha_{n_{\min}}, \text{ and } \alpha_i \leq \alpha'_i, \text{ for } i = 1, \dots, n'_{\min} \right\} \end{aligned} \quad (69)$$

being the feasible region; the exponent $\hat{E}(\alpha')$ in (68) is

$$\hat{E}(\alpha') = \min_{\alpha \in \mathcal{R}} E(\alpha', \alpha) \quad (70)$$

with $E(\alpha', \alpha) \triangleq E(\alpha' | \alpha) + E(\alpha)$. From (60) and (64),

$$\begin{aligned} E(\alpha', \alpha) &= \sum_{i=1}^{n'_{\min}} (n_{N+1} - i + 1) \alpha'_i + \sum_{j=1}^{n'_{\min}} \left((j - 1 - n_{N+1} + c_j) \alpha_j + \sum_{i < j} (\alpha'_i - \alpha_j)^+ \right) \\ &\quad + \sum_{j=n'_{\min}+1}^{n_{\min}} \left(c_j \alpha_j + \sum_{i=1}^{n'_{\min}} (\alpha'_i - \alpha_j)^+ \right). \end{aligned} \quad (71)$$

It remains to show that $\hat{E}(\alpha') = E'(\alpha') \triangleq \sum_i c_i \alpha'_i$ with

$$c'_i \triangleq 1 - i + \min_{k=1, \dots, N+1} \left\lfloor \frac{\sum_{l=0}^k \tilde{n}'_l - i}{k} \right\rfloor, \quad i = 1, \dots, n'_{\min} \quad (72)$$

by solving the optimization problem (70), which is accomplished in the rest of the section.

B. Solving the Optimization Problem

We need to distinguish three cases, according to how the value of n_{N+1} affects the ordered dimension \tilde{n}' .

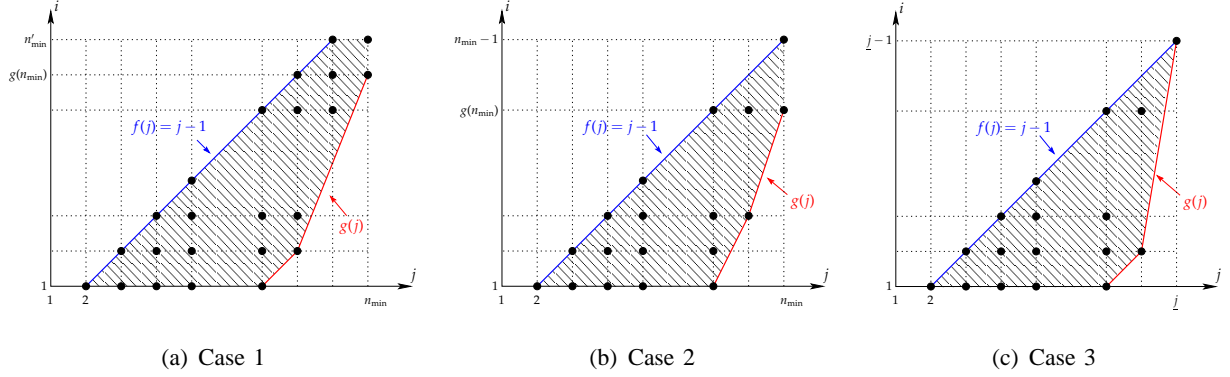


Fig. 7. For each j , the black dots represent the α'_i 's that are freed by α_j . Therefore, we can get the total number of freed α'_i by counting the black dots in row i . More precisely, there are $\lfloor g^{-1}(i) \rfloor - \lceil f^{-1}(i) \rceil + 1 = \lfloor g^{-1}(i) \rfloor - i$ black dots for $i \leq g(n_{\min})$, and $n_{\min} - \lceil f^{-1}(i) \rceil + 1 = n_{\min} - i$ black dots for $i > g(n_{\min})$.

1) *Case 1* [$n_{N+1} < \tilde{n}_0$]: In this case, we have $n'_{\min} = \tilde{n}'_0 = n_{N+1}$. Minimization of $E(\alpha, \alpha')$ of (71) with respect to α can be decomposed into n_{\min} minimizations with respect to $\alpha_1, \dots, \alpha_{n_{\min}}$ successively, i.e., $\min_{\alpha} = \min_{\alpha_{n_{\min}}} \cdots \min_{\alpha_1}$. We start with α_1 . From (61), the feasible region of α_1 is $0 \leq \alpha_1 \leq \alpha'_1$. Since the only α_1 -related term in (71) is $(c_1 - n_{N+1})\alpha_1$ and $c_1 - n_{N+1} > 0$ for $n_{N+1} < \tilde{n}_0$, we have $\alpha_1^* = 0$. Now, suppose that the minimization with respect to $\alpha_1, \dots, \alpha_{j-1}$ is done and that we would like to minimize with respect to α_j . For α_j , $j \leq n'_{\min}$, we set the initial region as

$$0 \leq \alpha'_1 \leq \cdots \leq \alpha'_{j-1} \leq \alpha_j \leq \alpha'_j$$

in which we have $\sum_{i < j} (\alpha'_i - \alpha_j)^+ = 0$. The feasibility conditions in (69) require that α_j must not go right across α'_j . The only choice is therefore to go to the left. Each time α_j goes across an α'_i from the right to the left, $(\alpha'_i - \alpha_j)^+$ increases by $\alpha'_i - \alpha_j$, which increases the coefficient of α'_i by 1 and decreases the coefficient of α_j by 1. It can be shown that, to minimize the value of $E(\alpha, \alpha')$ with respect to α_j , α_j is allowed to cross α'_i only when the current coefficient of α_j in (71) is positive.¹³ So, α_j stops moving only in the following two cases : 1) it hits the left extreme, 0; and 2) its coefficient achieves 0 when it is in the interval $[\alpha'_k, \alpha'_{k+1}]$ for some $k < j$. Either case, α_j -related terms are gone and what remain are the α'_i 's “freed” by α_j from $\sum_{i < j} (\alpha'_i - \alpha_j)^+$. Same reasoning applies to α_j for $j > n'_{\min}$, except that the initial region is set

¹³When the coefficient of α_i in (71) is positive, decreasing α_i decreases $E(\alpha, \alpha')$.

to $0 \leq \alpha'_1 \leq \dots \leq \alpha'_{n'_{\min}} \leq \alpha_j$.

Therefore, the optimization problem can be solved by counting the total number of freed α'_i 's. As shown in Fig. 7(a), when j is small, the initial coefficient of α_j is large and thus α_j can free out $\alpha'_{j-1}, \dots, \alpha'_1$. We have $\alpha_j^* = 0$, which corresponds to the first stopping condition. For large j , the initial coefficient of α_j is not large enough and only $\alpha'_{j-1}, \dots, \alpha'_{g(j)}$ is freed, which corresponds to the second stopping condition. With the above reasoning, we can get $g(j)$

$$g(j) = \begin{cases} j - 1 - (j - 1 - n_{N+1} + c_j) + 1, & \text{for } j \leq n'_{\min}, \\ n_{N+1} - c_j + 1, & \text{for } j > n'_{\min}. \end{cases} \quad (73)$$

From (73) and (14), we get

$$g(j) = n_{N+1} - \min_{k=1, \dots, N} \left\lfloor \frac{\sum_{l=0}^k \tilde{n}_l - (k+1)j}{k} \right\rfloor, \quad (74)$$

and

$$\lfloor g^{-1}(i) \rfloor = \min_{k=1, \dots, N} \left\lfloor \frac{\sum_{l=0}^k \tilde{n}_l - k(n_{N+1} - i)}{k+1} \right\rfloor. \quad (75)$$

Now, $\hat{E}(\alpha')$ can be obtained¹⁴ from Fig. 7(a)

$$\begin{aligned} \hat{E}(\alpha') &= \sum_{i=1}^{n'_{\min}} (n_{N+1} - i + 1) \alpha'_i + \sum_{i=1}^{g(n_{\min})} (\lfloor g^{-1}(i) \rfloor - i) \alpha'_i + \sum_{i=g(n_{\min})+1}^{n'_{\min}} (n_{\min} - i) \alpha'_i \\ &= \sum_{i=1}^{g(n_{\min})} (1 - 2i + n_{N+1} + \lfloor g^{-1}(i) \rfloor) \alpha'_i + \sum_{i=g(n_{\min})+1}^{n'_{\min}} (1 - 2i + n_{N+1} + n_{\min}) \alpha'_i \\ &= \sum_{i=1}^{g(n_{\min})} \left(1 - i + \min_{k=2, \dots, N+1} \left\lfloor \frac{\sum_{l=0}^k \tilde{n}'_l - i}{k} \right\rfloor \right) \alpha'_i + \sum_{i=g(n_{\min})+1}^{n'_{\min}} (1 - 2i + n_{N+1} + n_{\min}) \alpha'_i \end{aligned} \quad (76)$$

$$= \sum_{i=1}^{n'_{\min}} \left(1 - i + \min_{k=1, \dots, N+1} \left\lfloor \frac{\sum_{l=0}^k \tilde{n}'_l - i}{k} \right\rfloor \right) \alpha'_i \quad (77)$$

$$= E'(\alpha'), \quad (78)$$

where (76) is from (75) and the fact that $\tilde{n}'_0 = n_{N+1}$, $\tilde{n}'_l = \tilde{n}_{l-1}$, $l = 1, \dots, N+1$; (77) can be derived from lemma A2.1, since $p'_1 = n_{N+1} + \tilde{n}_0 - \tilde{n}_1 = g(n_{\min})$ and therefore the term \min_k

¹⁴In the above minimization procedure, we ignored the feasibility condition $\alpha_j \geq \alpha_k$, $\forall j > k$. A more careful analysis reveals that it is always satisfied with the described procedure.

in (77) is dominated by $k \geq 2$ for $i \leq g(n_{\min})$ and by $k = 1$ for $i > g(n_{\min})$, corresponding to the two terms in (76), respectively.

2) *Case 2* [$n_{N+1} \in [\tilde{n}_0, \tilde{n}_1]$]: In this case, we have $n'_{\min} = n_{\min}$ and $\tilde{n}'_1 = n_{N+1}$. From (71),

$$E(\alpha', \alpha) = \sum_{i=1}^{n'_{\min}} (n_{N+1} - i + 1) \alpha'_i + \sum_{j=1}^{n'_{\min}} \left((j - 1 - n_{N+1} + c_j) \alpha_j + \sum_{i < j} (\alpha'_i - \alpha_j)^+ \right). \quad (79)$$

Since $j - 1 - n_{N+1} + c_j > 0, \forall j \leq n'_{\min}$, the minimization of $E(\alpha', \alpha)$ with respect to α is in exactly the same manner as in the previous case. Therefore, $\hat{E}(\alpha')$ can be obtained from Fig. 7(b) with $g(j)$ in the same form as (74)

$$\begin{aligned} \hat{E}(\alpha') &= \sum_{i=1}^{n'_{\min}} (n_{N+1} - i + 1) \alpha'_i + \sum_{i=1}^{g(n_{\min})} (\lfloor g^{-1}(i) \rfloor - i) \alpha'_i + \sum_{i=g(n_{\min})+1}^{n'_{\min}} (n_{\min} - i) \alpha'_i \\ &= E'(\alpha'). \end{aligned} \quad (80)$$

3) *Case 3* [$n_{N+1} \in [\tilde{n}_1, \infty)$]: As in the previous case, we have $n'_{\min} = n_{\min}$ and the same $E(\alpha', \alpha)$ as defined in (79). Without loss of generality, we assume that $n_{N+1} \in [\tilde{n}_{k^*}, \tilde{n}_{k^*+1})$ for some $k^* \in [1, N]$ (we set $\tilde{n}_{N+1} \triangleq \infty$). Then, we have

$$\tilde{n}'_l = \tilde{n}_l, \quad \text{for } l = 1, \dots, k^*, \quad (81)$$

and

$$p_{k^*} < p'_{k^*} \leq p_{k^*-1} = p'_{k^*-1} \leq \dots \leq p_1 = p'_1. \quad (82)$$

Unlike the previous case, $j - 1 - n_{N+1} + c_j$ is not always positive. Let \underline{j} be the smallest integer such that the coefficient $j - 1 - n_{N+1} + c_j$ of α_j in (79) is zero. It is obvious that for $j \geq \underline{j}$, $\alpha_j^* = \alpha'_j$. Hence, we have

$$\hat{E}(\alpha') = \sum_{i=1}^{n'_{\min}} (n_{N+1} - i + 1) \alpha'_i + \sum_{i=1}^{\underline{j}-1} (\lfloor g^{-1}(i) \rfloor - i) \alpha'_i + \sum_{j=\underline{j}}^{n'_{\min}} (j - 1 - n_{N+1} + c_j) \alpha'_j,$$

where the second term is from Fig. 7(c). Furthermore, we can show that $\underline{j} \leq p'_{k^*}$, since $p'_{k^*} - 1 - n_{N+1} + c_{p'_{k^*}} = 0$. Therefore, we get

$$\hat{E}(\alpha') = \sum_{i=1}^{\underline{j}-1} (1 - 2i + n_{N+1} + \lfloor g^{-1}(i) \rfloor) \alpha'_i + \sum_{i=\underline{j}}^{p'_{k^*}-1} (n_{N+1} - i + 1) \alpha'_i + \sum_{i=p'_{k^*}}^{n'_{\min}} c_i \alpha'_i. \quad (83)$$

Now, we would like to show that the coefficient of α'_i in (83) coincides with c'_i . First, for $i \leq \underline{j}-1$, $i \in \mathcal{I}'_{k^*+1} \cup \dots \cup \mathcal{I}'_N$ and lemma A2.1 implies that

$$\begin{aligned} 1 - 2i + n_{N+1} + \lfloor g^{-1}(i) \rfloor &= 1 - i + \min_{k=2, \dots, N+1} \left\lfloor \frac{\sum_{l=0}^k \tilde{n}'_l - i}{k} \right\rfloor \\ &= 1 - i + \min_{k=1, \dots, N+1} \left\lfloor \frac{\sum_{l=0}^k \tilde{n}'_l - i}{k} \right\rfloor \\ &= c'_i. \end{aligned}$$

Then, for $i \geq p'_{k^*}$, we have

$$i \in (\mathcal{I}'_{k^*} \cup \dots \cup \mathcal{I}'_1) \cap (\mathcal{I}_{k^*} \cup \dots \cup \mathcal{I}_1).$$

Hence,

$$\begin{aligned} c'_i &= 1 - i + \min_{k=1, \dots, k^*} \left\lfloor \frac{\sum_{l=0}^k \tilde{n}'_l - i}{k} \right\rfloor \\ &= 1 - i + \min_{k=1, \dots, k^*} \left\lfloor \frac{\sum_{l=0}^k \tilde{n}_l - i}{k} \right\rfloor \\ &= c_i, \end{aligned} \tag{84}$$

where (84) is from (81) and (82). Finally, for $i \in [\underline{j}, p'_{k^*})$, let us rewrite $i = p'_{k^*} - \Delta_i$. Since $i - 1 - n_{N+1} + c_i = 0$, $\forall i \in [\underline{j}, p'_{k^*})$, we have

$$\begin{aligned} \left\lfloor \frac{\sum_{l=0}^{k^*} \tilde{n}_l - i - k^* n_{N+1}}{k^*} \right\rfloor &= \left\lfloor \frac{\sum_{l=0}^{k^*} \tilde{n}_l - p'_{k^*} + \Delta_i - k^* n_{N+1}}{k^*} \right\rfloor \\ &= \left\lfloor \frac{\Delta_i}{k^*} \right\rfloor \\ &= 0, \end{aligned}$$

from which we have $\Delta_i \in [0, k^* - 1]$ and

$$\begin{aligned} c'_i &= \left\lfloor \frac{\sum_{l=0}^{k^*} \tilde{n}_l + n_{N+1} - i}{k^* + 1} \right\rfloor + 1 - i \\ &= \left\lfloor \frac{\sum_{l=0}^{k^*} \tilde{n}_l + n_{N+1} - p'_{k^*} + \Delta_i}{k^* + 1} \right\rfloor + 1 - i \\ &= 1 + n_{N+1} - i. \end{aligned}$$

The proof is complete.

C. Proof of Proposition 3.1

Let $\alpha(\mathbf{M})$ denote the vector of the eigen-exponents of a matrix \mathbf{M} as previously defined. To prove the first case, we use induction on N . Suppose that it is true for N , which means that the joint pdf of $\alpha(\mathbf{\Pi}_g \mathbf{\Pi}_g^\dagger)$ is the same as that of $\alpha(\mathbf{\Pi} \mathbf{\Pi}^\dagger)$. Furthermore, we know by lemma A1.8 that $\alpha(\mathbf{\Pi}_g \mathbf{T}_{N,N+1} \mathbf{T}_{N,N+1}^\dagger \mathbf{\Pi}_g^\dagger) = \alpha(\mathbf{\Pi}_g \mathbf{\Pi}_g^\dagger)$. Same steps as (67)(68) complete the proof. To prove the second statement, we perform a singular value decomposition on the matrices $\mathbf{T}_{i,i+1}$'s and then apply the first statement.

APPENDIX III

PROOF OF THEOREM 3.2 AND THEOREM 3.3

A. Proof of Theorem 3.2

Let

$$c_i^{(m)} \triangleq 1 - i + \min_{k=1,\dots,m} \left\lfloor \frac{\sum_{l=0}^k \tilde{n}_l - i}{k} \right\rfloor, \quad i = 1, \dots, n_{\min}. \quad (85)$$

What we should prove is that $c_i^{(N)} = c_i^{(k)}$, for $i = 1, \dots, n_{\min}$ if and only if (16) is true. To this end, it is enough to show that

$$c_i^{(N)} = c_i^{(N-1)} \quad \text{for } i = 1, \dots, n_{\min} \quad (86)$$

if and only if $p_{N-1} \leq N - 1$, that is, $(N - 1)(\tilde{n}_N + 1) \geq \sum_{l=0}^{N-1} \tilde{n}_l$, and then apply the result successively to show the theorem. Note that we need Lemma A2.1 to eliminate the minimization in (85). The detailed proof is omitted here.

B. Proof of Theorem 3.3

The direct part of the theorem is trivial. To show the converse, let $\tilde{\mathbf{n}} \triangleq (\tilde{n}_0, \tilde{n}_1, \dots, \tilde{n}_N)$ and $\tilde{\mathbf{n}}' \triangleq (\tilde{n}'_0, \tilde{n}'_1, \dots, \tilde{n}'_{N'})$ be the two concerned minimal forms. In addition, we assume, without loss of generality, that

$$\begin{aligned} \tilde{n}_1 &= \dots = \tilde{n}_{i_1}, \dots, \tilde{n}_{i_{M-1}+1} = \dots = \tilde{n}_{i_M} \\ \tilde{n}'_1 &= \dots = \tilde{n}'_{i'_1}, \dots, \tilde{n}'_{i'_{M'}-1+1} = \dots = \tilde{n}'_{i'_{M'}} \end{aligned}$$

with $i_M \leq N$ and $i'_{M'} \leq N'$. Now, let us define $c_{0i} \triangleq c_i - (1 - i)$ with c_i defined in (66). It can be shown that M intervals are non-trivial with $|\mathcal{I}_{i_k}| \neq 0$, $k = 1, \dots, M$. The values of c_{0i} 's are in the following form

$$\overbrace{\dots, \tilde{n}_{i_M}, \dots, \tilde{n}_{i_M}}^{|I_{i_M}|}, \underbrace{\tilde{n}_{i_M} - 1, \dots, \tilde{n}_{i_M} - 1}_{i_{M-1}}, \dots, \underbrace{\tilde{n}_{i_{M-1}}, \dots, \tilde{n}_{i_{M-1}}}_{i_{M-1}}, \dots, \overbrace{\tilde{n}_2 - 1, \dots, \tilde{n}_1 + 1, \tilde{n}_1}^{|I_1|}.$$

Same arguments also apply to \tilde{n} with M' and i' , etc. It is then not difficult to see that to have exactly the same c_{0i} 's (thus, same c_i 's), we must have $N = N'$ and

$$\tilde{n}_i = \tilde{n}'_i, \quad \forall i = 0, \dots, N,$$

that is, the same minimal form.

APPENDIX IV

PROOF OF THEOREM 3.4

A. Sketch of the Proof

To prove the theorem, we will first show the following equivalence relations :

$$(R_1^{(N)}(k), R_3^{(N)}(i, k)) \xLeftrightarrow{(a)} (R_1^{(N)}(k), R_2^{(N)}(i)), \quad \forall i, k;$$

$$R_3^{(N)}(i, k) \xLeftrightarrow{(b)} R_3^{(N)}(N - 1, k), \quad \forall i, k;$$

$$(R_1^{(N)}(k), R_2^{(N)}(N - 1)) \xLeftrightarrow{(c)} (R_1^{(N)}(k), R_2^{(N)}(i) \text{ with ordered } \mathbf{n});$$

$$(R_1^{(N)}(k), R_2^{(N)}(i) \text{ with ordered } \mathbf{n}) \xLeftrightarrow{(d)} (R_1^{(N)}(k), R_2^{(N)}(N - 1) \text{ with ordered and minimal } \mathbf{n}).$$

1) *Equivalences (a) and (b)*: The direct parts of (a), (b), and (d) are immediate since the RHS are particular cases of the left hand side (LHS). To show the reverse part of (a), we rewrite

$$d_{(n_0, \dots, n_N)}^{\text{RP}}(k) = d_{(n_0 - k, \dots, n_N - k)}^{\text{RP}}(0) \tag{87}$$

$$= \min_{j \geq 0} \left\{ d_{(n_0 - k, \dots, n_i - k)}^{\text{RP}}(j) + d_{(j, n_{i+1} - k, \dots, n_N - k)}^{\text{RP}}(0) \right\} \tag{88}$$

$$= \min_{j' \geq k} \left\{ d_{(n_0, \dots, n_i)}^{\text{RP}}(j') + d_{(j', n_{i+1}, \dots, n_N)}^{\text{RP}}(k) \right\}, \tag{89}$$

where R_1 is used twice in (87) and (89); R_2 is used in (88). As for (b), if $R_3^{(N)}(N-1, k)$ holds, then

$$d_{(n_0, \dots, n_N)}^{\text{RP}}(k) = \min_{j \geq k} \left\{ d_{(n_0, \dots, n_{N-1})}^{\text{RP}}(j) + d_{(j, n_N)}^{\text{RP}}(k) \right\} \quad (90)$$

$$= \min_{j' \geq j \geq k} \left\{ d_{(n_0, \dots, n_{N-2})}^{\text{RP}}(j') + d_{(j', n_{N-1})}^{\text{RP}}(j) + d_{(j, n_N)}^{\text{RP}}(k) \right\} \quad (91)$$

$$= \min_{j' \geq k} \left\{ d_{(n_0, \dots, n_{N-2})}^{\text{RP}}(j') + d_{(j', n_{N-1}, n_N)}^{\text{RP}}(k) \right\} \quad (92)$$

which proves $R_3^{(N)}(N-2, k)$. By continuing the process, we can show that $R_3^{(N)}(i, k)$ is true for all i , provided $R_3^{(N)}(N-1, k)$ holds.

2) *Equivalences (c) and (d)*: Through (a) and (b), one can verify that the LHS of (c) is equivalent to the RHS of (a) of which the RHS of (c) is a particular case. Hence, the direct part of (c) is shown. The reverse part of (c) can be proved by induction on N . For $N = 2$, $R_2^{(N)}(N-1)$ can be shown explicitly using the direct characterization (13). Now, assuming that $R_2^{(N)}(N-1)$ for non-ordered \mathbf{n} , we would like to show that $R_2^{N+1}(N)$ holds. Let us write

$$\min_{j \geq 0} \left\{ d_{(n_0, \dots, n_N)}^{\text{RP}}(j) + d_{(j, n_{N+1})}^{\text{RP}}(0) \right\} = \min_{j \geq 0} \left\{ d_{(\tilde{n}_0, \dots, \tilde{n}_{i-1}, \tilde{n}_{i+1}, \dots, \tilde{n}_{N+1})}^{\text{RP}}(j) + d_{(j, \tilde{n}_i)}^{\text{RP}}(0) \right\} \quad (93)$$

$$= \min_{k \geq j \geq 0} \left\{ d_{(\tilde{n}_0, \dots, \tilde{n}_{i-1}, \tilde{n}_{i+1}, \dots, \tilde{n}_N)}^{\text{RP}}(k) + d_{(k, \tilde{n}_{N+1})}^{\text{RP}}(j) + d_{(j, \tilde{n}_i)}^{\text{RP}}(0) \right\} \quad (94)$$

$$= \min_{k \geq j' \geq 0} \left\{ d_{(\tilde{n}_0, \dots, \tilde{n}_{i-1}, \tilde{n}_{i+1}, \dots, \tilde{n}_N)}^{\text{RP}}(k) + d_{(k, \tilde{n}_i)}^{\text{RP}}(j') + d_{(j', \tilde{n}_{N+1})}^{\text{RP}}(0) \right\} \quad (95)$$

$$= \min_{j' \geq 0} \left\{ d_{(\tilde{n}_0, \dots, \tilde{n}_N)}^{\text{RP}}(j') + d_{(j', \tilde{n}_{N+1})}^{\text{RP}}(0) \right\} \\ = d_{(n_0, \dots, n_{N+1})}^{\text{RP}}(0),$$

where the permutation invariance property is used in (93); $R_3^{(N)}(N-1, k)$ is used in (94) since we assume that $R_2^{(N)}(N-1)$ is true; \tilde{n}_i and \tilde{n}_{N+1} can be permuted according to $R_2^{(2)}(1)$. Finally, we should prove the reverse part of (d), i.e.,

$$d_{(\tilde{n}_0, \dots, \tilde{n}_N)}^{\text{RP}}(0) = \min_{j \geq 0} \left\{ d_{(\tilde{n}_0, \dots, \tilde{n}_{N-1})}^{\text{RP}}(j) + j \tilde{n}_N \right\} \quad (96)$$

provided that $R_2^{(N)}(N-1)$ holds for minimal \mathbf{n} .

If \mathbf{n} is not minimal, then showing (c) is equivalent to showing

$$d_{(\tilde{n}_0, \dots, \tilde{n}_{N*})}^{\text{RP}}(0) = \min_{j \geq 0} \left\{ d_{(\tilde{n}_0, \dots, \tilde{n}_{N*})}^{\text{RP}}(j) + j \tilde{n}_N \right\}, \quad (97)$$

where N^* is the order of \mathbf{n} with $\tilde{n}_{N^*+1} \leq \tilde{n}_N$. Therefore, we should show that the minimum is achieved with $j = 0$. According the direct characterization (13), this is true only when $\tilde{n}_N \geq c_1$.

Let us rewrite c_1 as

$$\begin{aligned} c_1 &= \left\lfloor \frac{\sum_{l=0}^{N^*} \tilde{n}_l - 1}{N^*} \right\rfloor \\ &= \left\lfloor \frac{N^* \tilde{n}_{N^*+1} + p_{N^*} - 1}{N^*} \right\rfloor. \end{aligned}$$

Since $p_{N^*} \geq N^*$ is always true according to the reduction theorem, we have $c_1 \leq \tilde{n}_{N^*+1} \leq \tilde{n}_N$.

The rest of this section is devoted to proving that (96) holds for minimal \mathbf{n} .

B. Minimal \mathbf{n}

Now, we restrict ourselves in the case of minimal and ordered \mathbf{n} , i.e., we would like to prove

$$d_{(\tilde{n}_0, \dots, \tilde{n}_{N^*})}^{\text{RP}}(0) = \min_{j \geq 0} \left\{ d_{(\tilde{n}_0, \dots, \tilde{n}_{N^*-1})}^{\text{RP}}(j) + j \tilde{n}_N \right\}. \quad (98)$$

Since $c_{p_{N^*-1}} \leq \tilde{n}_{N^*}$, the optimal j is in the interval $\mathcal{I}_{N^*} \triangleq [1, p_{N^*-1}]$. Now, showing (98) is equivalent to showing

$$\sum_{i=1}^{p_{N^*-1}} 1 - i + \left\lfloor \frac{\sum_{l=0}^{N^*} \tilde{n}_l - i}{N^*} \right\rfloor = \min_{p_{N^*-1} \geq j \geq 0} \sum_{i=j+1}^{p_{N^*-1}} 1 - i + \left\lfloor \frac{\sum_{l=0}^{N^*-1} \tilde{n}_l - i}{N^* - 1} + j \tilde{n}_{N^*} \right\rfloor$$

which, after some simple manipulations, is reduced to

$$\sum_{i=1}^{p_M} \left(i - p_M + \left\lfloor \frac{i-1}{M+1} \right\rfloor \right) = \min_k \sum_{i=1}^k \left(i - p_M + \left\lfloor \frac{i-1}{M} \right\rfloor \right), \quad (99)$$

where we set $M \triangleq N^* - 1$ for simplicity of notation. Obviously, the minimum of the RHS of (99) is achieved with such k^* that

$$k^* - p_M + \left\lfloor \frac{k^* - 1}{M} \right\rfloor \leq 0, \quad (100)$$

$$\text{and } (k^* + 1) - p_M + \left\lfloor \frac{k^*}{M} \right\rfloor > 0. \quad (101)$$

Let us decompose k^* as $k^* = aM + b$ with $b \in [1, M]$. Then, (100) becomes

$$aM + b - p_M + a \leq 0 \quad (102)$$

which also implies that $aN + 1 - p_M + a \leq 0$ from which $a = \left\lfloor \frac{p_M - 1}{M+1} \right\rfloor$. The form of a suggests that p_M can be decomposed as

$$p_M = a(M+1) + \bar{b}. \quad (103)$$

From (102) and (103), we have $b \leq \bar{b}$ and thus $b = \min \{M, \bar{b}\}$. With the form of optimal k and some basic manipulations, we have finally

$$\sum_{i=1}^{p_M} \left(i - p_M + \left\lfloor \frac{i-1}{M+1} \right\rfloor \right) - \sum_{i=1}^{k^*} \left(i - p_M + \left\lfloor \frac{i-1}{M} \right\rfloor \right) = 0$$

which ends the proof.

APPENDIX V

PROOF OF LEMMAS 4.1 AND 4.2

A. Proof of Lemma 4.1

First, we have

$$\text{SNR} \lambda_{\max}(\mathbf{H}^\dagger \mathbf{H}) \leq \text{SNR} \|\mathbf{H}\|_{\text{F}}^2 \leq \det(\mathbf{I} + \text{SNR} \mathbf{H}^\dagger \mathbf{H}),$$

from which

$$\mathbf{P} \{ \text{SNR} \lambda_{\max}(\mathbf{H}^\dagger \mathbf{H}) < 1 + \epsilon \} \geq \mathbf{P} \{ \det(\mathbf{I} + \text{SNR} \mathbf{H}^\dagger \mathbf{H}) < 1 + \epsilon \} \quad (104)$$

with ϵ being some strictly positive constant. Then, we also have

$$\mathbf{P} \{ \text{SNR} \lambda_{\max}(\mathbf{H}^\dagger \mathbf{H}) < 1 + \epsilon \} \leq \mathbf{P} \{ \det(\mathbf{I} + \text{SNR} \mathbf{H}^\dagger \mathbf{H}) < (2 + \epsilon)^{\text{rank}(\mathbf{H})} \}, \quad (105)$$

since $\det(\mathbf{I} + \text{SNR} \mathbf{H}^\dagger \mathbf{H}) = \prod_i (\mathbf{I} + \text{SNR} \lambda_i(\mathbf{H}^\dagger \mathbf{H}))$. From (104) and (105), we have

$$\begin{aligned} \mathbf{P} \{ \text{SNR} \lambda_{\max}(\mathbf{H}^\dagger \mathbf{H}) < 1 + \epsilon \} &\doteq \mathbf{P} \{ \det(\mathbf{I} + \text{SNR} \mathbf{H}^\dagger \mathbf{H}) < 1 + \epsilon' \} \\ &\doteq \text{SNR}^{-d(0)}, \end{aligned}$$

where ϵ' is another strictly positive constant. Hence, $\mathbf{P} \{ \text{SNR} \|\mathbf{H}\|_{\text{F}}^2 < 1 + \epsilon \} \doteq \text{SNR}^{-d(0)}$. The lemma is proved since $\mathbf{P} \{ \text{SNR} \|\mathbf{H}\|_{\text{F}}^2 < 1 + \epsilon \} \doteq \mathbf{P} \{ \text{SNR} \|\mathbf{H}\|_{\text{F}}^2 < 1 \}$.

B. Proof of Lemma 4.2

Let us consider a parallel channel $\{\mathbf{H}_k\}_{k=1}^K$, each sub-channel of rank M_k and with eigen-exponents $\{\alpha_{1,k}, \alpha_{2,k}, \dots, \alpha_{M_k,k}\}$. Since each sub-channel is an AF path, the joint pdf of the eigen-exponents in the high SNR regime is $p_k(\boldsymbol{\alpha}_k) \doteq \text{SNR}^{-\sum_i c_{i,k} \alpha_{i,k}}$. From Lemma A1.1, the DMT is

$$d_{\mathcal{P}}(r) \triangleq \min_{\{\boldsymbol{\alpha}_k\}_{k \in \mathcal{O}(r)}} \sum_k \sum_i c_{i,k} \alpha_{i,k}$$

with $\mathcal{O}(r) \triangleq \{\sum_k \sum_i (1 - \alpha_{i,k})^+ \leq Kr\}$ being the outage region. First, we can deduce that

$$\begin{aligned} d_{\mathcal{P}}(0) &= \sum_k \sum_i c_{i,k} \\ &= \sum_k d_k(0). \end{aligned}$$

Then, if all AF paths have the same DMT, they have the same set $\{c_{i,k}\}_i$, i.e., $c_{i,k} = c_i, \forall k$. We can verify that setting $\alpha_{i,k} = \alpha_i, \forall k$ is without loss of optimality, since 1) the objective function is linear and symmetric on different k , and 2) the constraints are convex and symmetric on different k . Finally, the optimization problem becomes

$$\min_{\alpha \in \mathcal{O}_0(r)} K \sum_i c_i \alpha_i$$

with $\mathcal{O}_0(r) \triangleq \{\sum_i (1 - \alpha_i)^+ \leq r\}$ is the outage region of each single AF path. The lemma can be proved immediately from here.

APPENDIX VI

OTHER PROOFS

A. Proof of Proposition 4.2

Without loss of generality, we assume that $n_0 \geq n_2$. Then, the bottleneck of the channel is the $n_1 \times n_2$ channel. Since the partition achieves the maximum diversity, by theorem 4.1, the partition size is $K = K_1 K_2$ with the n_1 (respectively, n_2) antennas being partitioned into K_1 (respectively, K_2) supernodes. Moreover, for any AF path k in the partition, we have $n_{k,0} + 1 \geq n_{k,1} + n_{k,2}$. Adding all the K inequalities up gives

$$\sum_{k=1}^K n_{k,0} + K_1 K_2 \geq K_2 n_1 + K_1 n_2. \quad (106)$$

The sum in the LHS of (106) can be upper-bounded by $K_1 n_0$, since each supernode in the transmitter cannot be connected to more than K_1 nodes. Hence, we have the following inequality after some simple manipulations

$$K_1 \geq \left\lceil \frac{K_2 n_1}{K_2 + n_0 - n_2} \right\rceil,$$

from which we have the lower bound on the partition size

$$K_1 K_2 \geq K_2 \left\lceil \frac{K_2 n_1}{K_2 + n_0 - n_2} \right\rceil,$$

which is obviously increasing with K_2 . Therefore, the minimum lower bound is obtained by setting $K_2 = 1$ and it coincides with (29). It can be shown that this lower bound is achieved by partitioning the intermediate layer into K supernodes with K defined by (29) without partitioning either of the source and the destination antennas.

B. Proof of Theorem 4.2

Let us define the selection matrices $\mathbf{J}_{i,k}$'s as $n_i \times n_i$ diagonal matrices with

$$\mathbf{J}_{i,k}(j, j) = \begin{cases} 1 & \text{if } j \in \mathcal{S}_{i,k}, \\ 0 & \text{otherwise.} \end{cases}$$

First, we would like to prove that the maximum diversity gain is achieved. This can be done in two steps. The first step is to prove that the parallel channel $\{\mathbf{\Pi}_k''\}_k$ with $\mathbf{\Pi}_k'' \triangleq \mathbf{H}_N \prod_{i=1}^{N-1} (\mathbf{J}_{i,f_i(k)} \mathbf{H}_i)$ achieves the maximum diversity. To this end, note that by partitioning the rows (respectively, columns) of \mathbf{H}_N (respectively, \mathbf{H}_1) according to the indices in $\mathcal{S}_{N,1}, \dots, \mathcal{S}_{N,K_N}$ (respectively, $\mathcal{S}_{0,1}, \dots, \mathcal{S}_{0,K_1}$), the matrix $\mathbf{\Pi}_k''$ can be partitioned into $K_0 K_N$ blocks, each one being an AF path from the source to the destination. Therefore, $\{\mathbf{\Pi}_k''\}_k$ comprises $K_0 K_1 \dots K_N$ AF paths, i.e., all possible paths. Obviously, these paths include the K independent paths $\{\mathbf{\Pi}_k\}_k$ in the independent partition. Therefore, the maximum diversity is achieved since $\sum_{k=1}^{K'} \|\mathbf{\Pi}_k''\|_F^2 \geq \sum_{k=1}^K \|\mathbf{\Pi}_k\|_F^2$.

The key of the second step is to show that the set of matrices $\{\mathbf{\Pi}_k'\}_k$ defined in (30) is actually an invertible constant linear transformation of $\{\mathbf{\Pi}_k''\}_k$, i.e.,

$$\begin{bmatrix} \mathbf{\Pi}_1' & \dots & \mathbf{\Pi}_{K'}' \end{bmatrix} = \begin{bmatrix} \mathbf{\Pi}_1'' & \dots & \mathbf{\Pi}_{K'}'' \end{bmatrix} \mathbf{T}.$$

In this case, we have

$$\begin{aligned} \sum_{k=1}^{K'} \|\mathbf{\Pi}_k'\|_F^2 &\geq \lambda_{\min}(\mathbf{T}\mathbf{T}^\dagger) \sum_{k=1}^{K'} \|\mathbf{\Pi}_k''\|_F^2 \\ &\doteq \sum_{k=1}^{K'} \|\mathbf{\Pi}_k''\|_F^2 \end{aligned}$$

and the diversity is lower-bounded by the maximum diversity, according to Lemma 4.2. Hence, the FF scheme also achieves the maximum diversity. The key point is shown in the following. First, let us divide the set of indices $\{1, \dots, K'\}$ into K'/K_1 groups, each one comprising exactly K_1 integers i_1, \dots, i_{K_1} such that $f_j(i_1) = \dots = f_j(i_{K_1})$, $\forall j = 2, \dots, N-1$, and $f_1(i_j)$ varies

from 1 to K_1 . Then, we partition the set $\{\Pi'_k\}_k$ according to the partition of the indices described above. Hence, the matrices in the same group can be rewritten as $\{GF_{1,0}H_1, \dots, GF_{1,K_1}H_1\}$ with G being some matrix. We have

$$\begin{bmatrix} GF_{1,1}H_1 & \cdots & GF_{1,K_1}H_1 \end{bmatrix} = \begin{bmatrix} GJ_{1,1}H_1 & \cdots & GJ_{1,K_1}H_1 \end{bmatrix} T_1,$$

where T_1 is composed of $K_1 \times K_1$ blocks of matrices with the (i, j) -th block being $-\mathbf{I}$ if $i = j \geq 2$ and \mathbf{I} otherwise. We can verify that T_1 is invertible and with the transformation, the matrices $F_{1,k}$'s are replaced by $J_{1,k}$'s with the same indices. In the same manner, we can successively replace the matrices $F_{2,k}, \dots, F_{N-1,k}$ with $J_{2,k}, \dots, J_{N-1,k}$ by similar invertible transformations T_2, \dots, T_{N-1} as T_1 . Finally, we obtain $\{\Pi''_k\}_k$ and the total transformation is invertible, constant and linear.

Note that the parallel channel of the FF scheme is in outage for a target rate $K'r \log \text{SNR}$ implies that at least one of the sub-channels is in outage for a target rate $r \log \text{SNR}$. Therefore, one can show that $\text{SNR}^{-d^{\text{FF}}(r)} \leq \text{SNR}^{-d^{\text{AF}}(r)}$, from which $d^{\text{FF}}(r) \geq d^{\text{AF}}(r)$. Finally, by showing that $d^{\text{FF}}(r)$ is piece-wise linear with $K' \tilde{n}_0$ sections, we prove the theorem.

C. Proof of Theorem 5.2

Let $\lambda(M)$ and $\alpha(M)$ denote the vector of the ordered eigenvalues and the corresponding eigen-exponents of a matrix M . The theorem can be proved by showing a stronger result : the asymptotical pdf of $\alpha(\Pi_{\text{PF}}^\dagger \Pi_{\text{PF}})$ in the high SNR regime is identical to that of $\alpha(\Pi^\dagger \Pi)$. We show it by induction on N . For $N = 1$, since $\underline{H}_1 = H_1$, the result is direct. Suppose that the theorem holds for N . Let us show that it also holds for $N + 1$. Note that $\Pi'_{\text{PF}} = \underline{H}_{N+1} P_N \Pi_{\text{PF}} = \underline{H}_{N+1} \underline{D}_N \underline{Q}_N^\dagger \Pi_{\text{PF}}$, from which we have

$$\begin{aligned} (\Pi'_{\text{PF}})^\dagger \Pi'_{\text{PF}} &\sim \mathcal{W}_{n_0}(n_{N+1}, (\underline{D}_N \underline{Q}_N^\dagger \Pi_{\text{PF}})^\dagger (\underline{D}_N \underline{Q}_N^\dagger \Pi_{\text{PF}})) \\ &\sim \mathcal{W}_{\underline{n}_{\min}}(n_{N+1}, \lambda((\underline{D}_N \underline{Q}_N^\dagger \Pi_{\text{PF}})^\dagger (\underline{D}_N \underline{Q}_N^\dagger \Pi_{\text{PF}}))) \end{aligned}$$

for a given Π . Similarly, $\Pi'^\dagger \Pi' \sim \mathcal{W}_{n_{\min}}(n_{N+1}, \lambda(\Pi^\dagger \Pi))$. At high SNR, we can show that

$$\begin{aligned} \alpha((\underline{D}_N \underline{Q}_N^\dagger \Pi_{\text{PF}})^\dagger (\underline{D}_N \underline{Q}_N^\dagger \Pi_{\text{PF}})) &= \alpha((\underline{Q}_N^\dagger \Pi_{\text{PF}})^\dagger (\underline{Q}_N^\dagger \Pi_{\text{PF}})) \\ &= \alpha(\Pi_{\text{PF}}^\dagger \Pi_{\text{PF}}), \end{aligned}$$

where the first equality comes from lemma A1.8 and the second one holds because $(\underline{Q}_N^\dagger \Pi_{\text{PF}})^\dagger (\underline{Q}_N^\dagger \Pi_{\text{PF}}) = \Pi_{\text{PF}}^\dagger \Pi_{\text{PF}}$. Finally, since we suppose that the joint pdf of $\alpha((\Pi_{\text{PF}}^\dagger) \Pi_{\text{PF}})$ is the same as that of $\alpha(\Pi^\dagger \Pi)$, we can draw the same conclusion for $\alpha((\Pi_{\text{PF}}')^\dagger \Pi_{\text{PF}}')$ and $\alpha((\Pi')^\dagger \Pi')$.

D. Proof of Theorem 6.1

Let us consider an equivalent block-diagonal channel of the parallel channel (43) in the following form

$$\mathbf{y}_e = \text{diag}(\Pi_k) \mathbf{x}_e + \mathbf{z}_e, \quad (107)$$

where $\mathbf{x}_e \triangleq [\mathbf{x}_1^\top \mathbf{x}_2^\top \dots \mathbf{x}_K^\top]^\top$, and $\mathbf{y}_e, \mathbf{z}_e$ are defined in the same manner. Now, from the parallel NVD code \mathcal{X} , we can build a block-diagonal code \mathcal{X}_{BD} with codewords defined by $\mathbf{X}_{\text{BD}} \triangleq \text{diag}\{\mathbf{X}_k\}$. We can verify that \mathcal{X}_{BD} is actually a rate- n_{av} NVD code defined in [34] with $n_{\text{av}} \triangleq \sum_k n_{t,k}/K$. From [34, Th. 3], we have

$$\begin{aligned} d_{\mathcal{X}_{\text{BD}}}(r) &\geq d\left(\frac{\sum_k n_{t,k}}{n_{\text{av}}} r\right) \\ &= d(K r), \end{aligned}$$

where $d(r)$ is the DMT of the parallel channel (and thus the block-diagonal channel). Finally, it is obvious that $d_{\mathcal{X}}(K r) = d_{\mathcal{X}_{\text{BD}}}(r)$, since using \mathcal{X} will have the same error performance¹⁵ as using \mathcal{X}_{BD} except that the transmission rate is K times higher. We have thus $d_{\mathcal{X}}(r) \geq d(r)$. It is shown in [34] that the achievability holds for any fading statistics. Thus, the code is approximately universal.

E. An Alternative Code Construction

A simple alternative construction that is approximately universal is described as follows. Let $\mathcal{X}_{\text{full}}$ be a $n_{\text{sum}} \times T$ full rate NVD code with $n_{\text{sum}} \triangleq \sum_k n_{t,k}$ and $T \geq n_{\text{sum}}$. Then, $\mathcal{X}_{\text{full}}$ achieves the DMT $d(r)$ of the channel (107). It means that by partitioning every codeword matrix $\mathbf{X}_{\text{full}} \in \mathcal{X}_{\text{full}}$ into $K \times 1$ blocks in such a way that the k th block is of size $n_{t,k} \times T$ and sending the k th block in the k th sub-channel, the DMT of the original parallel channel is achieved. Although this construction is simple and suitable for both symmetric and asymmetric channels, the main drawback is that the coding delay is roughly K times larger than the parallel NVD code constructed in Section VI-C. Decoding complexity of such codes is sometimes prohibitive.

¹⁵This is due to the block-diagonal nature of the equivalent channel.

F. ζ_{2^m} is not a norm in \mathbb{K}

Assume that ζ_{2^m} is a norm in \mathbb{K} , which means

$$\exists x \in \mathbb{K}, N_{\mathbb{K}/\mathbb{Q}(\zeta_{2^m})}(x) = \zeta_{2^m}. \quad (108)$$

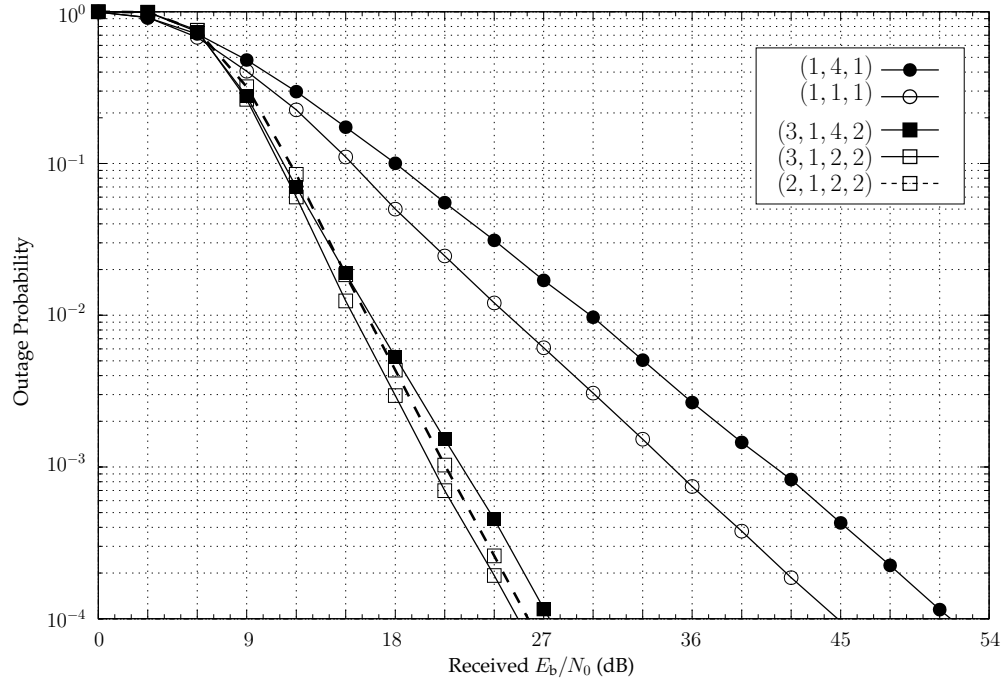
Consider now the extensions described in Fig. 6 with the proper fields. From (108) and the left extension of Fig. 6, we deduce that $N_{\mathbb{K}/\mathbb{Q}(i)}(x) = N_{\mathbb{Q}(\zeta_{2^m})/\mathbb{Q}(i)}(N_{\mathbb{K}/\mathbb{Q}(\zeta_{2^m})}(x)) = -i$, since the minimal polynomial of ζ_{2^m} is $X^{2^{m-2}} - i$. Meanwhile, from the right extension of Fig. 6, we have $N_{\mathbb{K}/\mathbb{Q}(i)}(x) = N_{\mathbb{Q}(i, \sqrt{5})/\mathbb{Q}(i)}(N_{\mathbb{K}/\mathbb{Q}(i, \sqrt{5})}(x)) = -i$. Denote $y = N_{\mathbb{K}/\mathbb{Q}(i, \sqrt{5})}(x) \in \mathbb{Q}(i, \sqrt{5})$. Then the number $z = \frac{1+\sqrt{5}}{2}y$ has an algebraic norm equal to i , and belongs to $\mathbb{Q}(i, \sqrt{5})$ which is in contradiction with the result obtained in [48]. So, ζ_{2^m} is a non-norm element.

REFERENCES

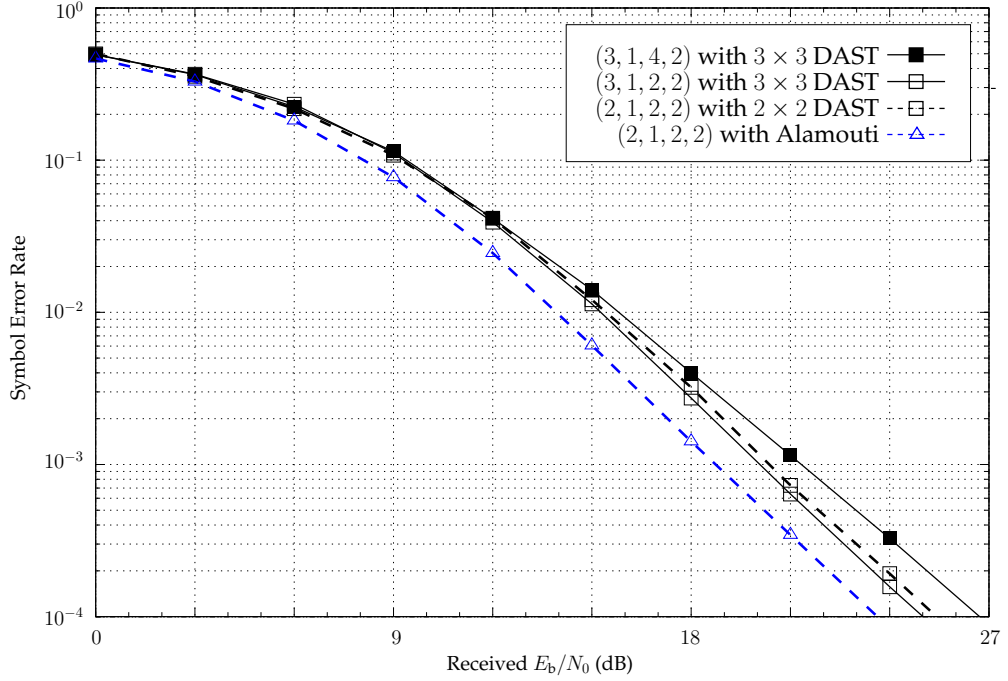
- [1] E. C. van der Meulen, "Three-terminal communication channels," *Adv. Appl. Prob.*, vol. 3, no. 1, pp. 120–154, 1971.
- [2] T. M. Cover and A. El Gamal, "Capacity theorems for the relay channel," *IEEE Trans. Inf. Theory*, vol. 25, no. 5, pp. 572–584, Sep. 1979.
- [3] G. Kramer, M. Gastpar, and P. Gupta, "Cooperative strategies and capacity theorems for relay networks," *IEEE Trans. Inf. Theory*, vol. 51, no. 9, pp. 3037–3063, Sep. 2005.
- [4] B. Wang, J. Zhang, and A. Høst-Madsen, "On the capacity of MIMO relay channels," *IEEE Trans. Inf. Theory*, vol. 1, no. 1, pp. 29–43, Jan. 2005.
- [5] P. Gupta and P. R. Kumar, "The capacity of wireless networks," *IEEE Trans. Inf. Theory*, vol. 46, no. 2, pp. 388–404, Mar. 2000.
- [6] L. Xie and P. R. Kumar, "A network information theory for wireless communication : Scaling laws and optimal operation," *IEEE Trans. Inf. Theory*, vol. 50, no. 5, pp. 748–767, May 2004.
- [7] M. Gastpar and M. Vetterli, "On the capacity of large Gaussian relay networks," *IEEE Trans. Inf. Theory*, vol. 51, no. 3, pp. 765–779, Mar. 2005.
- [8] V. I. Morgenshtern and H. Bölcskei, "Crystallization in wireless networks," *IEEE Trans. Inf. Theory*, vol. 53, no. 10, Oct. 2007, to appear.
- [9] A. Özgür, O. Lévêque, and D. N. C. Tse, "Hierarchical cooperation achieves optimal capacity scaling in ad hoc networks," *IEEE Trans. Inf. Theory*, Sep. 2006, submitted.
- [10] A. Sendonaris, E. Erkip, and B. Aazhang, "User cooperation diversity—Part I: System description," *IEEE Trans. Commun.*, vol. 51, no. 11, pp. 1927–1938, Nov. 2003.
- [11] —, "User cooperation diversity—Part II: Implementation aspects and performance analysis," *IEEE Trans. Commun.*, vol. 51, no. 11, pp. 1939–1948, Nov. 2003.
- [12] J. N. Laneman and G. W. Wornell, "Distributed space-time-coded protocols for exploiting cooperative diversity in wireless networks," *IEEE Trans. Inf. Theory*, vol. 49, no. 10, pp. 2415–2425, Oct. 2003.
- [13] J. N. Laneman, D. N. C. Tse, and G. W. Wornell, "Cooperative diversity in wireless networks: Efficient protocols and outage behavior," *IEEE Trans. Inf. Theory*, vol. 50, no. 12, pp. 3062–3080, Dec. 2004.

- [14] T. Hunter, S. Sanayei, and A. Nosratinia, "Outage analysis of coded cooperation," *IEEE Trans. Inf. Theory*, vol. 52, no. 2, pp. 375–391, Feb. 2006.
- [15] R. U. Nabar, H. Bölcskei, and F. W. Kneubühler, "Fading relay channels: Performance limits and space-time signal design," *IEEE J. Sel. Areas Commun.*, vol. 22, no. 6, pp. 1099–1109, Aug. 2004.
- [16] K. Azarian, H. El Gamal, and P. Schniter, "On the achievable diversity-multiplexing tradeoff in half-duplex cooperative channels," *IEEE Trans. Inf. Theory*, vol. 51, no. 12, pp. 4152–4172, Dec. 2005.
- [17] S. Yang and J.-C. Belfiore, "Towards the optimal amplify-and-forward cooperative diversity scheme," *IEEE Trans. Inf. Theory*, vol. 53, no. 9, Sep. 2007, to appear. [Online]. Available: <http://arxiv.org/pdf/cs.IT/0603123>
- [18] M. Yuksel and E. Erkip, "Multi-antenna cooperative wireless systems: A diversity multiplexing tradeoff perspective," *IEEE Trans. Inf. Theory*, Oct. 2007, special Issue on Models, Theory, and Codes for Relaying and Cooperation in Communication Networks, to appear.
- [19] H. Bölcskei, R. U. Nabar, Ö. Oyman, and A. J. Paulraj, "Capacity scaling laws in MIMO relay networks," *IEEE Trans. Wireless Commun.*, vol. 5, no. 6, pp. 1433–1444, Jun. 2006.
- [20] Y. Jing and B. Hassibi, "Distributed space-time coding in wireless relay networks," *IEEE Trans. Wireless Commun.*, vol. 5, no. 12, pp. 3524–3536, Dec. 2006.
- [21] —, "Cooperative diversity in wireless relay networks with multiple-antenna nodes," *IEEE Trans. Signal Process.*, 2006, submitted.
- [22] S. Yeh and O. Lévêque, "Asymptotic capacity of multi-level amplify-and-forward relay networks," in *Proc. IEEE International Symposium on Information Theory*, Nice, France, Jun. 2007.
- [23] L. Zheng and D. N. C. Tse, "Diversity and multiplexing: A fundamental tradeoff in multiple-antenna channels," *IEEE Trans. Inf. Theory*, vol. 49, no. 5, pp. 1073–1096, May 2003.
- [24] D. N. C. Tse and P. Viswanath, "Diversity-multiplexing tradeoff in multiple access channels," *IEEE Trans. Inf. Theory*, vol. 50, no. 9, pp. 1859–1874, Sep. 2004.
- [25] T. M. Cover and J. Thomas, *Elements of Information Theory*. New York: Wiley, 1991.
- [26] F. Oggier, G. Rekaya, J.-C. Belfiore, and E. Viterbo, "Perfect space-time block codes," *IEEE Trans. Inf. Theory*, vol. 52, no. 9, pp. 3885–3902, Dec. 2006.
- [27] P. Elia, K. R. Kumar, S. A. Pawar, P. V. Kumar, and H. Lu, "Explicit, minimum-delay space-time codes achieving the diversity-multiplexing gain tradeoff," *IEEE Trans. Inf. Theory*, vol. 52, no. 9, pp. 3869–3884, Sep. 2006.
- [28] S. Tavildar and P. Viswanath, "Approximately universal codes over slow fading channels," *IEEE Trans. Inf. Theory*, vol. 52, no. 7, pp. 3233–3258, Jul. 2006.
- [29] S. Borade, L. Zheng, and R. Gallager, "Amplify and forward in wireless relay networks: Rate, diversity and network size," *IEEE Trans. Inf. Theory*, Oct. 2007, special Issue on Relaying and Cooperation in Communication Networks, to appear.
- [30] R. R. Müller, "On the asymptotic eigenvalue distribution of concatenated vector-valued fading channels," *IEEE Trans. Inf. Theory*, vol. 48, no. 7, pp. 2086–2091, Jul. 2002.
- [31] P. Wolniansky, G. Foschini, G. Golden, and R. Valenzuela, "V-BLAST: An architecture for realizing very high data rates over the rich-scattering wireless channel," in *Proc. of the URSI International Symposium on Signal, Systems, and Electronics Conference*, New York, 1998, pp. 295–300.
- [32] F. Oggier and B. Hassibi, "An algebraic coding scheme for wireless relay networks with multiple-antenna nodes," *IEEE Trans. Signal Process.*, Mar. 2006, submitted.

- [33] P. Elia and P. V. Kumar, "Approximately universal optimality over several dynamic and non-dynamic cooperative diversity schemes for wireless networks." [Online]. Available: <http://fr.arxiv.org/pdf/cs.IT/0512028>
- [34] S. Yang and J.-C. Belfiore, "Optimal space-time codes for the MIMO amplify-and-forward cooperative channel," *IEEE Trans. Inf. Theory*, vol. 53, no. 2, pp. 647–663, Feb. 2007.
- [35] R. A. Horn and C. R. Johnson, *Matrix Analysis*. New York: Cambridge, 1985.
- [36] B. A. Sethuraman, B. S. Rajan, and V. Shashidhar, "Full-diversity, high-rate space-time block codes from division algebras," *IEEE Trans. Inf. Theory*, vol. 49, no. 10, pp. 2596–2616, Oct. 2003.
- [37] S. Yang, J.-C. Belfiore, and G. Rekaya-Ben Othman, "Perfect space-time block codes for parallel MIMO channels," in *Proc. IEEE International Symposium on Information Theory*, Seattle, WA, Jul. 2006.
- [38] F. Oggier and E. Viterbo, "Algebraic number theory and code design for Rayleigh fading channels," in *Foundations and Trends in Communications and Information Theory*, 2004, vol. 1, no. 3, pp. 333–415.
- [39] E. Bayer-Fluckiger, F. Oggier, and E. Viterbo, "New algebraic constructions of rotated \mathbb{Z}^n -lattice constellations for the rayleigh fading channel," *IEEE Trans. Inf. Theory*, vol. 50, no. 4, pp. 702–714, Apr. 2004.
- [40] M. O. Damen, K. Abed-Meraim, and J.-C. Belfiore, "Diagonal algebraic space time block codes," *IEEE Trans. Inf. Theory*, vol. 48, no. 3, pp. 628–636, March 2002.
- [41] S. Alamouti, "Space-time block coding: A simple transmitter diversity technique for wireless communications," *IEEE J. Sel. Areas Commun.*, vol. 16, pp. 1451–1458, Oct. 1998.
- [42] A. Edelman, "Eigenvalues and condition numbers of random matrices," Ph.D. Dissertation, MIT, 1989.
- [43] A. T. James, "Distributions of matrix variates and latent roots derived from normal samples," *Annals of Math. Statistics*, vol. 35, pp. 475–501, 1964.
- [44] H. Gao and P. J. Smith, "A determinant representation for the distribution of quadratic forms in complex normal vectors," *J. Multivariate Analysis*, vol. 73, pp. 155–165, May 2000.
- [45] S. H. Simon, A. L. Moustakas, and L. Marinelli, "Capacity and character expansions: Moment generating function and other exact results for MIMO correlated channels," *IEEE Trans. Inf. Theory*, vol. 52, no. 12, pp. 5336–5351, Dec. 2006.
- [46] A. M. Tulino and S. Verdu, "Random matrix theory and wireless communications," in *Foundations and Trends in Communications and Information Theory*, 2004, vol. 1, no. 1, pp. 1–182.
- [47] S. Yang and J.-C. Belfiore, "Diversity-multiplexing tradeoff of double scattering MIMO channels," *IEEE Trans. Inf. Theory*, Mar. 2006, submitted for publication. [Online]. Available: <http://arxiv.org/pdf/cs.IT/0603124>
- [48] J.-C. Belfiore, G. Rekaya, and E. Viterbo, "The Golden code: A 2×2 full-rate space-time code with non-vanishing determinants," *IEEE Trans. Inf. Theory*, vol. 51, no. 4, pp. 1432–1436, Apr. 2005.

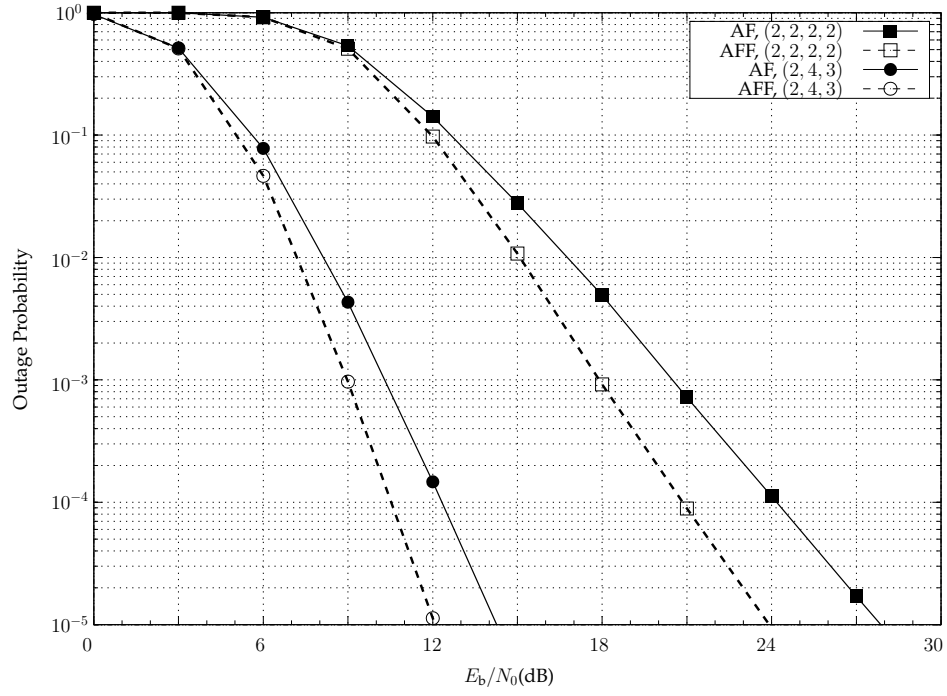


(a) Outage probability

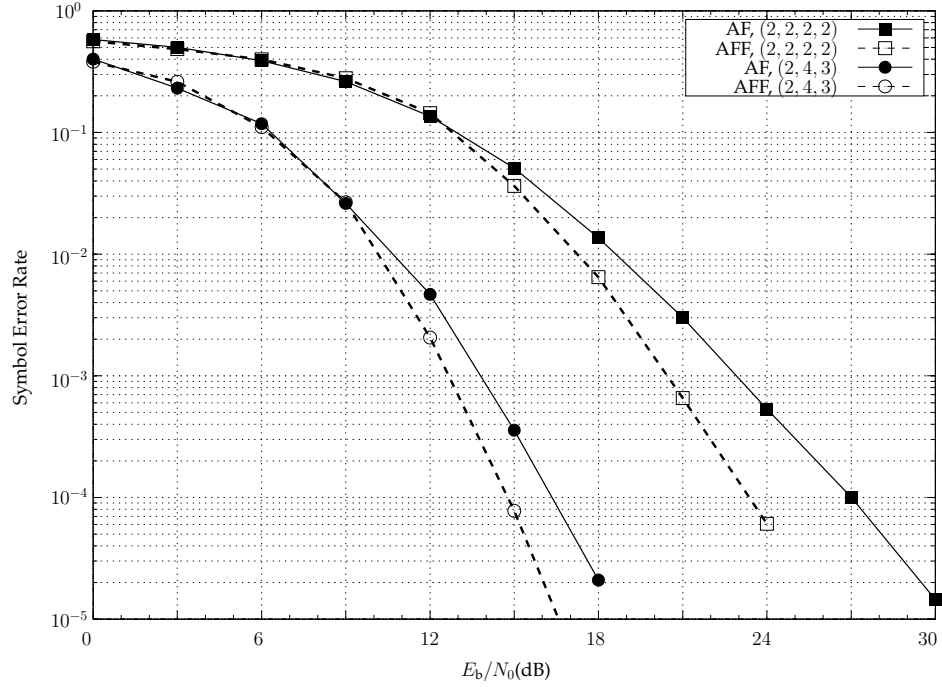


(b) Symbol error rate

Fig. 8. Vertical reduction : target data rate 2 bits per channel use in the outage performances or 4-QAM constellation in the coded cases.



(a) Outage probability



(b) Symbol error rate

Fig. 9. AF vs. AFF : target data rate 4 bits per channel use in the outage performances or 4-QAM constellation in the coded cases.

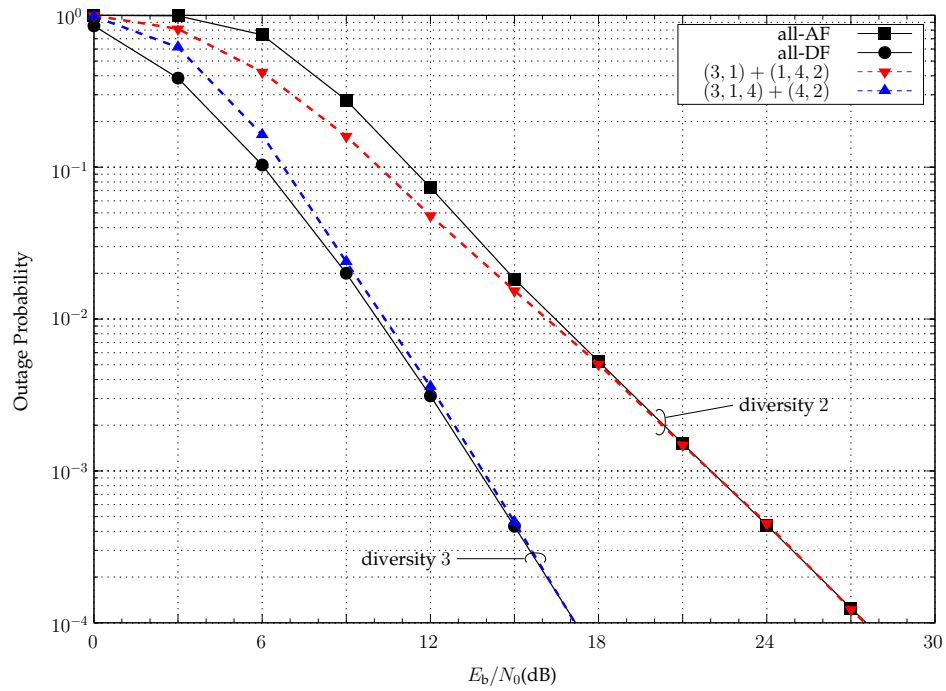


Fig. 10. The $(3, 1, 4, 2)$ multihop channel : outage probability of the serial partition with various numbers of decoding clusters, target data rate 2 bits per channel use.